On-line appendix to Detecting and Analyzing the Effects of Time-Varying Parameters in DSGE Models

Fabio Canova, Norwegian Business School, and CEPR
Filippo Ferroni, Federal Reserve Bank of Chicago
Christian Matthes, Federal Reserve Bank of Richmond *

April 15, 2019

Abstract

We study how structural parameter variations affect the decision rules and economic inference. We provide diagnostics to detect parameter variations and to ascertain whether they are exogenous or endogenous. A constant parameter model poorly approximates a time-varying DGP, except in a handful of relevant cases. Linear approximations do not produce time-varying decision rules; higher order approximations can do this only if parameter disturbances are treated as decision rule coefficients. Structural responses are time invariant regardless of order of approximation. Adding endogenous variations to the parameter controlling leverage in Gertler and Karadi's (2010) model substantially improves the fit of the model.

Key words: Structural model, time-varying parameters, endogenous variations, misspecification.

JEL Classification: C10, E27, E32.

*We thank Jesus Fernandez Villaverde (the editor), two anonymous referees, Stephane Bonhomme, Francesco Bianchi, Ferre de Graeve, Marco del Negro, James Hamilton, Lars Hansen, Michele Lenza, Frank Schorfheide, Harald Uhlig, and Tao Zha as well as participants of many seminars and conferences for their comments and suggestions. Canova acknowledges the financial support from the Spanish Ministerio de Economia y Competitividad through the grants ECO2012-33247; ECO2015-68136-P; and FEDER, UE. The views presented in this paper are not necessarily those of the Federal Reserve Bank of Richmond, the Federal Reserve Bank of Chicago, or the Federal Reserve System. Earlier versions of this paper circulated under the title "Approximating Time Varying Structural Models with Time Invariant Structures."

Appendix A: Proofs and derivations

Proposition 1

Proof. Substituting the linear decision rule into the optimality conditions, we obtain

$$\begin{aligned}0 =&[FP^2 + (G + N\phi_x)P + (H + O\phi_x)]x_{t-1} + [(FP + G + N\phi_x)Q + FQ\psi_z + L\psi_z + M]z_t \\ &+ [(FP + G + N\phi_x)R + FR\omega_u + N\phi_u\omega_u + O\phi_u]u_t\end{aligned}$$

Since the solution must hold for every realization of x_{t-1}, z_t, u_t, we need to equate their coefficient to zero and the result obtains. □

Derivation of second order decision rules

When parameters are time-varying, the approximate second order law of motion of the structural parameters is

$$\theta_t = \phi_x x_{t-1} + \phi_u u_t + 1/2\ \Delta\ \lambda_t \tag{1}$$

where

$$\lambda_t = vec\left(\begin{bmatrix} x_{t-1} \\ u_t \end{bmatrix} \begin{bmatrix} x'_{t-1} & u'_t \end{bmatrix}\right)$$

$\Delta = \begin{bmatrix} \phi'_1 \\ \vdots \\ \phi'_{n_\theta} \end{bmatrix}$; $\phi_j = vec\left(\begin{bmatrix} \phi^j_{xx} & \phi^j_{ux} \\ \phi^j_{ux} & \phi^j_{uu} \end{bmatrix}\right)$, and $\phi^j_{xx} = \left\{\frac{\partial^2 \theta_{jt}}{\partial x_{ht} \partial x_{it}}\right\}$. The approximate equilibrium conditions are:

$$0 = E_t(Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t + N\theta_{t+1} + O\theta_t + 1/2\Gamma^1_{xz\theta}\Lambda_{t+1} + 1/2\Gamma^0_{xz\theta}\Lambda_t) \tag{2}$$

$$\Lambda_t = vec\left(\begin{bmatrix} x_t \\ x_{t-1} \\ z_t \\ \theta_t \end{bmatrix} \begin{bmatrix} x'_t & x'_{t-1} & z'_t & \theta'_t \end{bmatrix}\right)$$

and $\Gamma^i_{xz\theta},\ i = 0, 1$ contains the coefficients on the cross derivative terms. Let

$$\widetilde{\Lambda}_t = vec\left(\begin{bmatrix} x_t \\ x_{t-1} \\ z_t \\ u_t \end{bmatrix} \begin{bmatrix} x'_t & x'_{t-1} & z'_t & u'_t \end{bmatrix}\right)$$

Notice that $\lambda_t = (J_0 \otimes J_0)\ \widetilde{\Lambda}_t$ and $\Lambda_t = (J_1 \otimes J_1)\ \widetilde{\Lambda}_t$ where

$$J_0 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix};\ J_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & \phi_x & 0 & \phi_u \end{bmatrix}$$

Using (1), and disregarding terms of order higher than two, the optimality conditions can be rewritten as:

$$0 = E_t\{Fx_{t+1} + (G + N\phi_x)x_t + (H + O\phi_x)x_{t-1} + [L \;\; M]\begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} + [N\phi_u \;\; O\phi_u]\begin{bmatrix} u_{t+1} \\ u_t \end{bmatrix}$$
$$+\frac{1}{2}\;\; (N\Delta(J_0 \otimes J_0) + \Gamma^1_{xz\theta}(J_1 \otimes J_1)) \;\; \widetilde{\Lambda}_{t+1} + \frac{1}{2}\;\; (O\Delta(J_0 \otimes J_0) + \Gamma^0_{xz\theta}(J_1 \otimes J_1)) \;\; \widetilde{\Lambda}_t \;\}$$
$$(3)$$

Following Lombardo and Sutherland (2007), one can rewrite (3) as

$$0 = E_t\left(Fx_{t+1} + (G + N\phi_x)x_t + (H + O\phi_x)x_{t-1} + [L \;\; M]\begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} + [N\phi_u \;\; O\phi_u]\begin{bmatrix} u_{t+1} \\ u_t \end{bmatrix} + \frac{1}{2}\;\; \mathbf{A}\;\widetilde{\Lambda}_t\right) + \mathbf{B} \qquad (4)$$

where

$$\mathbf{A} = (N\Delta(J_0 \otimes J_0) + \Gamma^1_{xz\theta}(J_1 \otimes J_1))(\widetilde{P} \otimes \widetilde{P}) + (O\Delta(J_0 \otimes J_0) + \Gamma^0_{xz\theta}(J_1 \otimes J_1))$$
$$\mathbf{B} = 1/2 \;\; (N\Delta(J_0 \otimes J_0) + \Gamma^1_{xz\theta}(J_1 \otimes J_1)) \;\; (\widetilde{Q} \otimes \widetilde{Q}) \;\; \Sigma$$

and $\tilde{P} = \begin{bmatrix} P & 0 & Q\phi_z & R\omega_u \\ I & 0 & 0 & 0 \\ 0 & 0 & \phi_z & 0 \\ 0 & 0 & 0 & \omega_u \end{bmatrix}$; $\tilde{Q} = \begin{bmatrix} Q & R \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$. The solution to (4) is given by

$$x_t = Px_{t-1} + Qz_t + Ru_t + \mathbf{C}\widetilde{\Lambda}_t + \mathbf{D} \qquad (5)$$

where, by construction, P, Q, R are the same as in the first order solution, C solves $S\mathbf{C} = -vec(\frac{1}{2}\mathbf{A})$, $S = I_\ell \otimes (FP + G + N\phi_x)$, $\ell = 2n_x + n_z + n_u$ and $\mathbf{D}$ is a function of $1/2\; (N\Delta(J_0 \otimes J_0) + \Gamma^1_{xz\theta}(J_1 \otimes J_1))\; (\widetilde{Q} \otimes \widetilde{Q})\Sigma$.

When the model has constant coefficients, the optimality condition (3) is

$$0 = E_t\left(Fx_{t+1} + Gx_t + Hx_{t-1} + [L \;\; M]\begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} + [0 \;\; 0]\begin{bmatrix} u_{t+1} \\ u_t \end{bmatrix} + \frac{1}{2}\;\; \mathbf{A}^{cc}\;\widetilde{\Lambda}_t\right) + \mathbf{B}^{cc} \qquad (6)$$

where

$$\mathbf{A}^{cc} = \Gamma^1_{xz\theta}(J_1^{cc} \otimes J_1^{cc})(\widetilde{P^{cc}} \otimes \widetilde{P^{cc}}) + \Gamma^0_{xz\theta}(J_1^{cc} \otimes J_1^{cc})$$
$$\mathbf{B}^{cc} = 1/2 \;\; \Gamma^1_{xz\theta}(J_1^{cc} \otimes J_1^{cc}) \;\; (\widetilde{Q}^{cc} \otimes \widetilde{Q^{cc}}) \;\; \Sigma$$

$$\tilde{P}^{cc} = \begin{bmatrix} P^{cc} & 0 & Q^{cc}\phi_z & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & \phi_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \tilde{Q}^{cc} = \begin{bmatrix} Q^{cc} & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix} J_0^{cc} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; J_1^{cc} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the terms in $\Gamma^1_{xz\theta}$ corresponding to the second order or cross derivatives with respect to θ_t are zero. The solution in this case is

$$x_t = P^{cc}x_{t-1} + Q^{cc}z_t + \mathbf{C}^{cc}\widetilde{\Lambda}_t + \mathbf{D}^{cc} \qquad (7)$$

where, C^{cc} solves $S\mathbf{C}^{cc} = -vec(\frac{1}{2}\mathbf{A^{cc}})$, and $\mathbf{D}^{cc}$ is a function of B^{cc}.

Proposition 4

Proof. In this case, $\phi_x = 0$ $\phi_{xx} = \phi_{xu} = 0$, $\phi_u \neq 0$ $\phi_{uu} \neq 0$. Thus, the solutions to (4) and (6) differ only because of terms involving u_t, $u_t u_t'$, and its variance. $\square$

Appendix B: The models of section 4 and additional Monte Carlo evidence

In the capacity utilization model, the representative household maximizes:

$$\max E_0 \sum_{j=0}^{\infty} \beta^t \frac{c_t^{1-\eta}}{1-\eta} - A\frac{n_t^{1+\gamma}}{1+\gamma} \tag{8}$$

$$\begin{aligned} c_t + i_t &= w_t n_t + r_t^k k_t^s - a(u_t)k_t^s - T_t && (9)\\ i_t &= k_t - (1-\delta)k_{t-1} && (10)\\ k_t^s &= u_t k_{t-1} && (11) \end{aligned}$$

where c_t is consumption, i_t investment, k_t the stock of capital, and n_t is hours worked. Household chooses the utilization rate of capital, u_t, and the amount of effective capital that she can rent to the firm, i.e., k_t^s. Household receives earnings from supplying labor and capital services to the firm, i.e., w_t and r_t^k, respectively, subject to a cost of changing capital utilization, $a(u_t)k_t^s$. Finally, T_t are lump-sum taxes levied by the government. The production function is $y_t = z_t(k_t^s)^\alpha n_t^{1-\alpha}$ A fraction of output is consumed by the government and financed with lump-sum taxes. The government budget constrain is $g_t y_t = T_t$.

The optimality conditions of the planner problem are

$$\begin{aligned} (1-g_t)y_t &= c_t + k_t - (1-\delta - a(u_t)u_t)\ k_{t-1} && (12)\\ An_t^\gamma c_t^\eta &= (1-\alpha)(1-g_t)y_t/n_t && (13)\\ 1 &= \beta E_t(c_t/c_{t+1})^\eta(1-\delta - a(u_{t+1})u_{t+1} + \alpha(1-g_{t+1})y_{t+1}/k_t) && (14)\\ \alpha(1-g_t)y_t/k_{t-1} &= u_t(a'(u_t)u_t + a(u_t)) && (15)\\ y_t &= z_t(u_t k_{t-1})^\alpha n_t^{1-\alpha} && (16)\\ a(u_t) &= \frac{1/\beta + \delta - 1}{\phi}\left(e^{\phi(u_t-1)} - 1\right) && (17) \end{aligned}$$

(17) gives the functional form for the adjustment cost of the capital utilization. Note that $a(1) = 0$, $a'(1) = 1/\beta + \delta - 1$, $a''(1) = \phi(1/\beta + \delta - 1)$. If assume that, in the steady state $u = 1$, the steady states for $(\frac{k}{y}, \frac{c}{y}, \frac{n}{y}, n)$ are the same as in the RBC model without variable capital utilization.

The RBC model with one period time to build is as in the text except the capital accumulation and production function equations are substituted by

$$K_t = (1-\delta)K_{t-1} + i_{t-1} \tag{18}$$

$$Y_t = \zeta_t K_{t-2}^{\alpha} N_t^{1-\alpha} \tag{19}$$

When stochastic volatility is present the variance of the technological disturbance is

$$\log \sigma_{z,t} = 0.01 \log \sigma_{z,0} + 0.99 \log \sigma_{z,t-1} + v_t \tag{20}$$

where $\log \sigma_{z,0} = 0.00712$ and $\mathrm{var}(v_t) = 0.01$.

For the RBC model with capital adjustment cost we assume

$$K_t = (1-\delta)K_{t-1} + i_{t-1} + 2(k_t - k_{t-1})^2 \tag{21}$$

In the model with two states, we use data simulated from constant parameter model and the baseline parameterization for 40 periods; for the remaining periods the data comes from the baseline model when $\delta = 0.015; \gamma = 1.0; \eta = 1.0; E(z) = 1.5$.

In the model with occasionally binding constraints, we assume that investment cannot fall below a certain exogenous threshold in every period,

$$i_t \geq \Psi i_{ss}, \tag{22}$$

and derive policy functions using Guerrieri and Iacoviello's (2015) algorithm.

DGP	Estimated model	Optimality wedge		Forecast errors	
		T=500	T=150	T=500	T=150
Fixed parameter	Fixed parameter	0.00	0.00	0.00	0.00
Exogenous parameter	Fixed parameter	0.90	0.78	1.00	0.99
Endogenous parameter	Fixed parameter	0.99	0.61	1.00	0.99
Endogenous parameter (internalization)	Fixed parameter	0.96	0.80	1.00	0.99

Table B.1: Percentage of rejections at the 0.05 confidence level of the null of no time variations in 200 experiments. The dependent variable is either the Euler wedge or the output forecast error. The regressors are lagged consumption and lagged real rate for the Euler wedge; lagged output, lagged consumption, and lagged hours for the forecast error. Parameter variations explain 15-20 cents of output variance

Appendix C: Inferential distortions

Identification

We are concerned with whether population identification pathologies described by Canova and Sala (2009) may also emerge as a byproduct of time-varying misspecification. We ask whether parameters that could be identified if the correct likelihood is employed can became poorly identified when the incorrect likelihood is used. Magnusson and Mavroedis (2014) have shown that when a GMM approach is used, time variations in certain parameters help the identification of time-invariant parameters. Huang (2014) qualifies the result by showing that time variations in weakly identified parameters have no effects on the asymptotic distribution of strongly identified parameters.

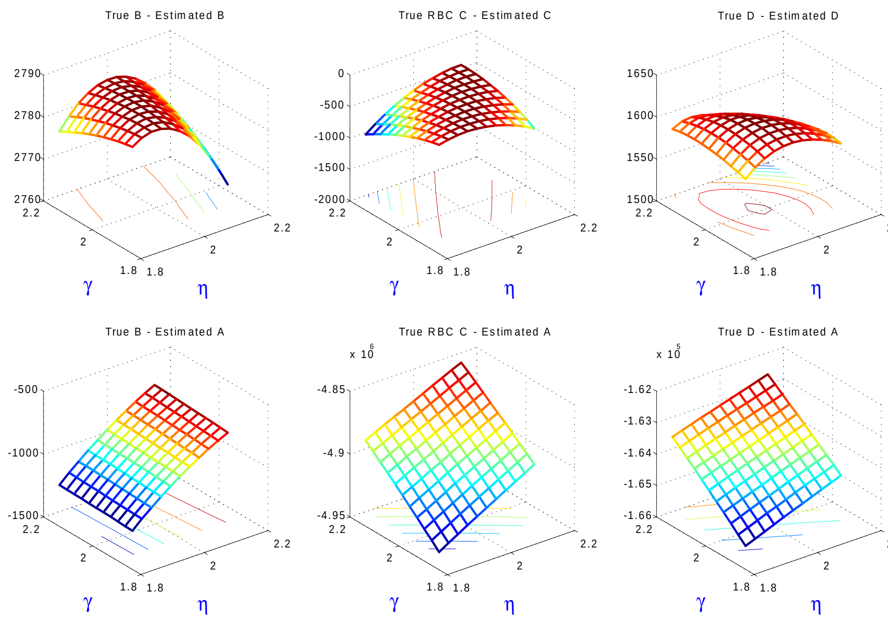

Figure C.1: Likelihood surfaces, RBC model

Figure C.1 plots the likelihood function of the RBC model of section 2 in the risk-aversion coefficient η and the inverse elasticity of labor supply γ when the forecast errors of the correct linearized model (top row) and of the constant parameter linearized model (bottom row) are used to construct the likelihood function. The first column considers data generated by the model B (exogenously varying parameters), the second by model C (endogenously varying parameters, but no internalization), the third by model D (endogenously varying parameters and internalization). The other parameters are fixed as in section 2.5. While the likelihood curvature in the correct model is not large, the maximum can be found at $\gamma = 2$, $\eta = 2$ for all three specifications (see top row). When the decision rules of the constant parameter model are used to construct the likelihood, the distortions are large: with exogenous parameter variations, γ becomes weakly identified; with endogenous specifications, the likelihood is locally convex, there is a ridge in (γ, η) - we can identify the sum of the two but not each of them separately - and the maximum is shifted away from the true values.

Note that shock misaggregation is present in all three columns, while decision rule misspecification is present only in columns 2 and 3. Thus, while both induce identification pathologies, the distortions created by the latter are larger.

A formal identification analysis conducted using Koop et al. (2013) statistic (see table C.1) confirms the graphical conclusions. Koop et al. show that asymptotically the precision matrix grows at the rate T for identified parameters and at rate less than T for underidentified parameters. Thus, the precision of the estimates, scaled by the sample size, converges to a constant for identified parameters and to zero for underidentified parameters. Furthermore, the magnitude of the constant measures identification strength: a large value indicates a strongly identified parameter; a small value a weakly identified one.

Parameter	T=150	T=300	T=500	T=750	T=1000	T=1500	T=2500	
	DGP Model B, Estimated model A							
η	15.9	17.8	17.2	18.8	18.4	19.3	17.9	
γ	28.5	45.7	108.4	81.4	93.6	104.2	90.17	
ρ_z	1.8e+4	2.6e+4	4.2e+4	4.2e+4	4.5e+4	4.9e+4		
ρ_g	209.2	655.5	2741	2190	2860	3417	2802	
δ	927.3	973.8	1.7e+4	1.7e+4	2.4e+4	2.3e+4		
α	140.2	156.2	264.2	215.5	239.1	252.1	229.3	
A	28.42	30.67	7.99	10.99	9.15	7.83	9.83	
	DGP Model C, Estimated model A							
η	822	1033	743	785	759	746	752	
γ	2261	3147	2682	2809	2720	2579	2566	
ρ_z	3073	2673	2952	2909	2799	2806	2877	
ρ_g	1.74	2.23	2.44	2.96	3.17	2.82	2.90	
δ	4.6e+5	4.4e+5	4.3e+5	4.0e+5	3.8e+5	4.4e+5		
α	1.8e+4	1.1e+4	1.4e+4	1.2e+4	1.1e+4	1.6e+4		
A	351	493	441	505	500	449	444	
	DGP Model D, Estimated model A							
η	550	575	592	610	545	542	494	
γ	3577	2442	2660	2870	2564	2711	2430	
ρ_z	1613	1243	1120	1162	1068	1189	1074	
ρ_g	1.22	1.28	1.44	1.53	1.60	1.62	1.67	
δ	5.2e+5	6.7e+5	6.5e+5	6.0e+5	5.7e+5	5.8e+5		
α	1.1e+4	2.5e+4	2.4e+4	1.9e+4	2.1e+4	2.0e+4		
A	488	276	340	382	349	395	334	

Table C.1: Koop, Pesaran, and Smith diagnostic. Reported are the diagonal elements of the precision matrix scaled by the sample size

When the DGP is model B and a fixed parameter model is considered, all parameters are identified. When the DGP are models C and D, all parameters but ρ_g seem identifiable. Interestingly, in models C and D, ρ_g is weakly identified, even when the correct likelihood is used. Thus, time variations in β_t and δ_t do not help in the identification of ρ_g, in line with Huang (2014).

Structural estimation

To study the properties of likelihood-based estimates of a misspecified constant parameter model, we generate data from the linearized versions of the RBC models B, C, D; estimate the structural parameters using the likelihood function constructed with the linear decision rules of the time-invariant model, flat priors, and standard Montecarlo Markov Chain (MCMC) methods; and repeat the exercise 150 times using different shock realizations. To benchmark the size of the distortions, we also estimate the structural parameters using the likelihood constructed with the correct decision rules (i.e., model B rules if the data is generated with model B, etc.).

The parameterization we employ is as in section 2.5. For the time-varying parameters, we set $d_t = \beta_{t+1}/\beta_t$ and assume that, in model B, $\Theta_{t+1} - \Theta \equiv (d_{t+1} - \beta, \delta_{t+1} - \delta)'$ $= U_{t+1}$, where $\beta = 0.99, \delta = 0.025$, and the components of $U_{t+1} = (u_{d,t+1}, u_{\delta,t+1})'$ are independent AR(1) with persistence $\rho_d = 0.9, \rho_\delta = 0.8$ and standard deviation $\sigma_d = 0.008, \sigma_\delta = 0.01$, respectively. For the other two models, the time-varying parameters evolve according to $\Theta_{t+1} = [\Theta_u - (\Theta_u - \Theta_l) \circ e^{-\phi_a(K_t - K)}] + [\Theta_u - (\Theta_u - \Theta_l) \circ e^{\phi_b(K_t - K)}] + U_{t+1}$, where $\Theta_u' = (0.999, 0.025)$; also for model C, $\phi_a' = (0.01, 0.03)$, $\phi_b' = (0.2, 0.1)$, U_{t+1} is i.i.d. with $\Sigma_u = diag(0.008, 0.005)$, while for model D, $\phi_a' = (0.001, 0.016)$, $\phi_b' = (0.2, 0.1)$, U_{t+1} is i.i.d. with $\Sigma_u = diag(0.009, 0.001)$. With these choices parameter variations explain 3-6 percent of the forecast error variance of output; we regard this as a conservative choice. We discuss also what happens when shocks to the parameters are more important.

Table C.2 contains a summary of the results: we report the time invariant parameters used to generate the data (column 1); the median of the distribution of posterior estimates obtained with the correct decision rules (column 2); the median, the 5th, and the 95th percentiles of the distribution of posterior estimates obtained with the decision rules of the time invariant model, when $T = 150$ (columns 3-5) and when $T = 1000$ (columns 6-8).

Figure C.2 plots the density of posterior estimates in the three cases. When the model is correctly specified, the density of posterior estimates should cluster around the true value. Thus, if the median is away from the true parameter value and/or the spread of the distribution is large, likelihood-based methods have difficulties in recovering the constant parameters of the data-generating process.

True Parameter	Correct Median	Constant 5th percentile	Parameter Median	Model 95th percentile	Constant 5th percentile	Parameter Median	Model 95th percentile
	T=150	T=150			T=1000		
$\eta = 2.0$	2.01	0.87	2.08	2.16	0.87	1.38	2.00
$\gamma = 2.0$	2.00	2.31	2.55	2.75	2.13	2.53	2.67
$\rho_z = 0.9$	0.89	0.93	0.94	0.98	0.94	0.95	0.98
$\rho_g = 0.5$	0.49	0.80	0.86	0.89	0.74	0.85	0.87
$\delta = 0.025$	0.02	0.01	0.01	0.01	0.01	0.01	0.03
$\alpha = 0.3$	0.30	0.10	0.15	0.18	0.10	0.13	0.17
$\eta = 2.0$	1.98	0.55	1.38	2.00	0.55	1.50	2.00
$\gamma = 2.0$	2.00	1.62	1.97	2.47	1.62	1.98	2.50
$\rho_z = 0.9$	0.90	0.91	0.95	1.00	0.91	0.97	1.00
$\rho_g = 0.5$	0.49	0.51	0.85	1.00	0.51	0.85	1.00
$\delta = 0.025$	0.02	0.02	0.07	0.07	0.04	0.07	0.08
$\alpha = 0.3$	0.30	0.17	0.25	0.28	0.19	0.26	0.29
$\eta = 2.0$	1.56	0.39	0.61	1.84	0.04	0.37	1.30
$\gamma = 2.0$	1.98	1.52	2.23	2.58	0.79	1.32	2.30
$\rho_z = 0.9$	0.90	0.91	0.99	1.00	0.94	1.00	1.00
$\rho_g = 0.5$	0.60	0.99	1.00	1.00	1.00	1.00	1.00
$\delta = 0.025$	0.03	0.01	0.02	0.03	0.01	0.01	0.03
$\alpha = 0.3$	0.30	0.10	0.15	0.22	0.10	0.13	0.22

Table C.2. Summary of the distributions of posterior estimates.

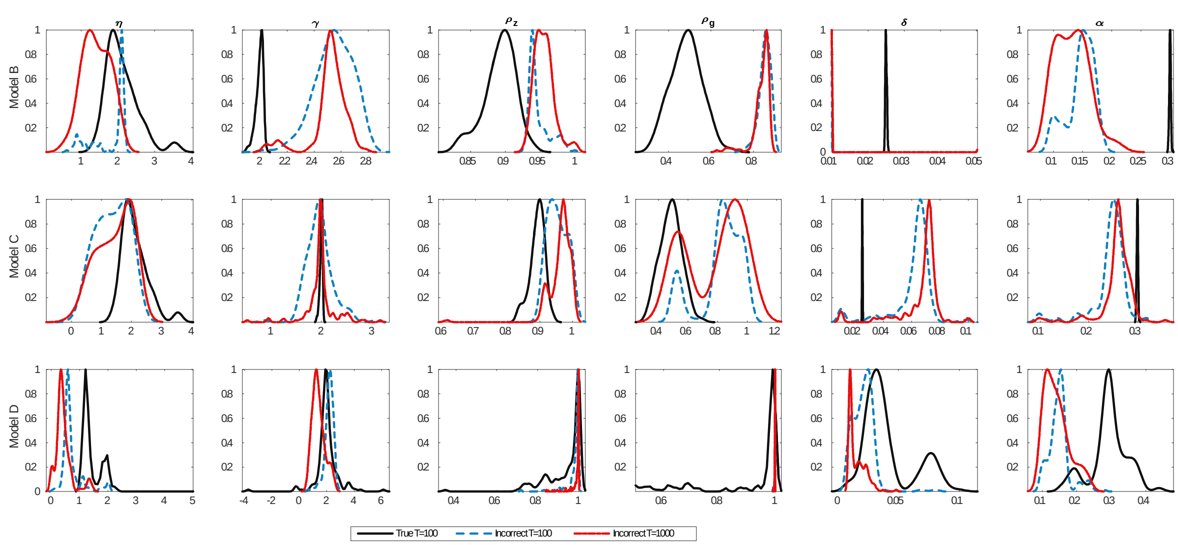

Figure C.2: Density of estimates. Parameter variations explain 3-6 percent of output variance).

Figure C.3 plots impulse responses to technology shocks: in each box we have the responses obtained using the values reported in column 2 of table C.2 together with the 16th and 84th percentiles of the distribution of responses obtained using the estimated distribution of parameters produced by the time invariant model when $T = 150$. Table C.3 shows the long-run variance decomposition, when $T = 150$ and the median of the estimated posterior distribution is used. In the first two columns we have the contribution of technology and government spending shocks to output, consumption, hours, and capital variability in the correct model; in the last two columns we have their contribution when the constant parameter model is used.

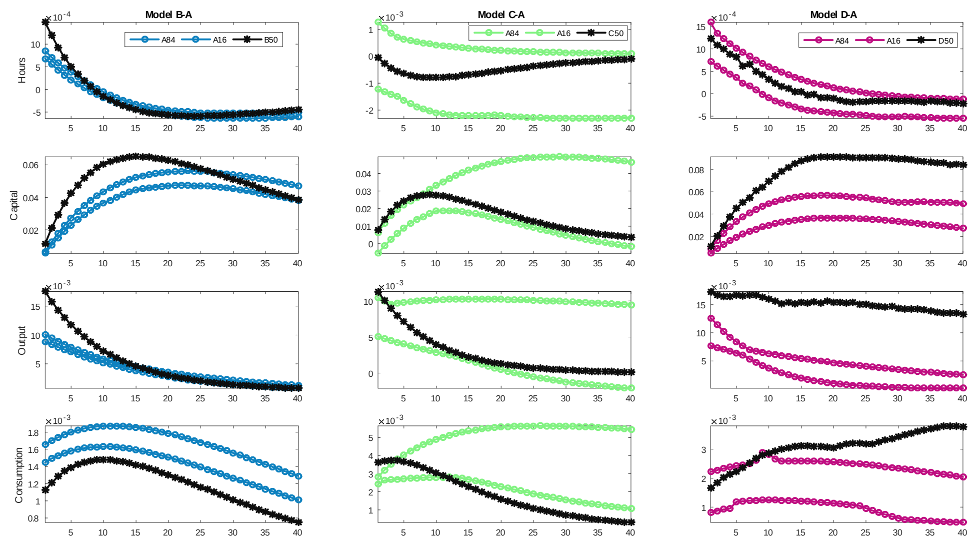

Figure C.3: Impulse responses to technology shocks

	Variance attributed to			
	Technology Shocks	Government Shocks	Technology Shocks	Government Shocks
	DGP: Model B			
	True		Estimated Time invariant	
Y	0.942	0.002	0.968	0.032
C	0.791	0.045	0.586	0.414
N	0.478	0.068	0.376	0.624
K	0.749	0.058	0.564	0.436
	DGP: Model C			
	True		Estimated Time invariant	
Y	0.965	0.006	0.977	0.023
C	0.780	0.055	0.878	0.122
N	0.430	0.139	0.499	0.501
K	0.592	0.108	0.949	0.051
	DGP: Model D			
	True		Estimated: Time invariant	
Y	0.898	0.002	0.738	0.262
C	0.836	0.055	0.439	0.561
N	0.393	0.153	0.573	0.427
K	0.829	0.128	0.757	0.243

Table C.3 Long-run variance decomposition

A number of features of table C.2 are worth discussing. When the correct model is employed, the estimation is successful, regardless of the DGP and of whether time variations are exogenous (model B) or externally endogenous (model C). Some distortions estimating η and ρ_g occur when model D is the DGP, but numerical biases are small. A number of distortions occur when a time-invariant model is used in estimation. For example, when exogenous variations are present, the persistence of government spending shocks is poorly estimated (mean persistence is about 50 percent larger than in the DGP), while estimates of (δ, α) are severely biased downward. The distortions become larger when time variations are endogenous (models C and D), and a significant upward bias exists in both ρ_z and ρ_g. Because in models C and D, parameter variations are serially correlated, a time-invariant model can capture these variations only by increasing the persistence of structural shocks. Note also that the performance of the time-invariant model is invariant to the sample size.

Responses to technology shocks are generally off in terms of impact magnitude and responses produced with estimates of the true model tend to be outside the estimated 68 percent band produced with estimates of the time-invariant model. Misspecification is larger when endogenous time variations exist.

What is the contribution of structural disturbances to the variability of the endogenous variables? One should expect the structural shocks obtained with the time invariant-model to be contaminated because the wrong P matrix is used to compute forecast errors and because we are aggregating m (primitive and parameter) shocks into $n < m$ (structural) shocks making them function of lags (and possible leads) of all original disturbances (see, e.g., Canova and Paustian, 2011). Indeed, important biases emerge. When models B or C are the DGP, the relative contribution of government spending shocks to the forecast error variance of hours, capital, and consumption is greatly overestimated. When the DGP is model D, this happens also for the forecast error variance of output.

When parameter variations explain a larger portion of the forecast error variance of output, the distortions are large (tables C.4-C.5 and figures C.4-C.5). For example, the density of posterior parameter estimates is often bimodal and skewed, and at times the density of estimates obtained in the time invariant model hardly overlaps with the density of estimates obtained with the correct model.

We have also performed a Monte Carlo exercise allowing the labor share to be time-varying. Variations in the labor share have been documented in, e.g., Rios Rull and Santaeularia Llopis (2010), and there is evidence that they are strongly countercyclical. This is relevant for our exercise because all four optimality conditions are affected by time variations, altering the strength of income and substitution effect distortions. Indeed, we do find that the distortions become quite large, and in many cases it becomes difficult to estimate the time-invariant model, regardless of the DGP (results available on request).

In sum, estimating a constant parameter model when the linearized DGP features time-varying parameters leads to identification and inferential distortions. This is true regardless of the sample size, of whether variations are exogenous or endogenous, and of whether parameter variations account for a small or a larger portion of output variability.

True Parameter	Correct Model Median	Constant 5th percentile	Coefficient Median	Model 95th percentile	Constant 5th percentile	Coefficient Median	Model 95th percentile
	T=150	T=150			T=1000		
$\eta = 2.0$	2.05	0.42	1.01	2.04	0.38	0.52	1.14
$\gamma = 2.0$	2.00	1.62	2.25	2.60	1.88	2.43	2.60
$\rho_z = 0.9$	0.89	0.90	0.97	1.00	0.96	1.00	1.00
$\rho_g = 0.5$	0.48	0.79	0.94	1.00	0.93	0.98	1.00
$\delta = 0.025$	0.02	0.01	0.03	0.09	0.01	0.01	0.03
$\alpha = 0.3$	0.30	0.10	0.18	0.33	0.10	0.12	0.17
$\eta = 2.0$	2.00	1.12	2.00	2.01	1.50	2.00	2.01
$\gamma = 2.0$	2.00	1.77	2.00	2.11	1.79	2.02	2.05
$\rho_z = 0.9$	0.90	0.90	0.92	0.97	0.90	0.91	0.95
$\rho_g = 0.5$	0.50	0.45	0.52	1.00	0.44	0.51	0.98
$\delta = 0.025$	0.03	0.02	0.06	0.07	0.05	0.06	0.07
$\alpha = 0.3$	0.30	0.17	0.23	0.26	0.20	0.22	0.26
$\eta = 2.0$	1.99	1.70	2.06	2.10	0.15	2.03	5.72
$\gamma = 2.0$	2.00	1.40	2.01	2.05	0.06	0.37	2.02
$\rho_z = 0.9$	0.97	0.85	0.90	0.97	0.74	0.96	0.99
$\rho_g = 0.5$	0.51	1.00	1.00	1.00	1.00	1.00	1.00
$\delta = 0.025$	0.03	0.01	0.08	0.09	0.01	0.01	0.07
$\alpha = 0.3$	0.30	0.10	0.28	0.30	0.15	0.22	0.25

Table C.4: Distributions of posterior estimates. Parameter variations explain 15-20 percent of output variance.

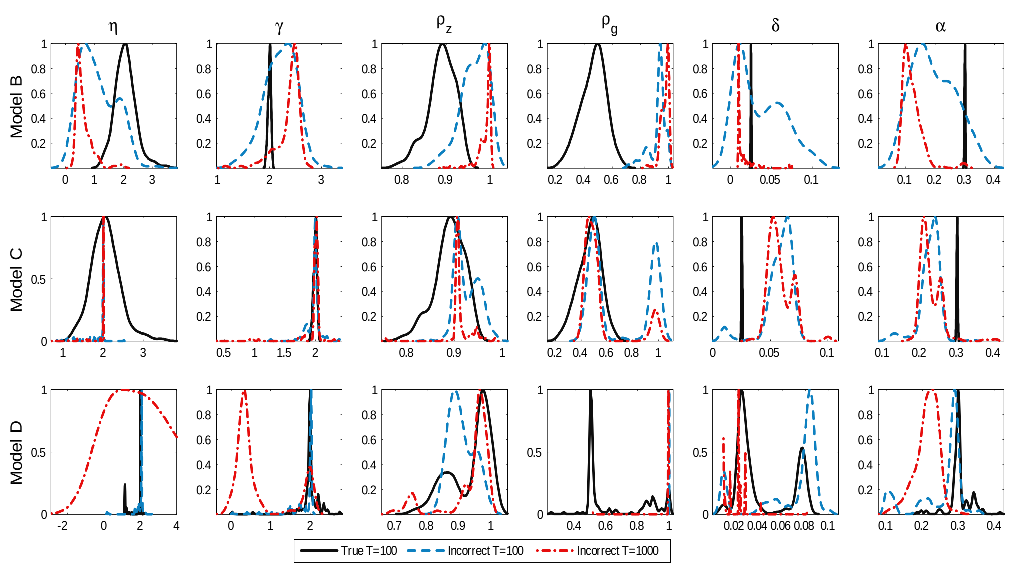

Figure C.4: Density of estimates. Parameter variations explain 15-20 percent of output variance

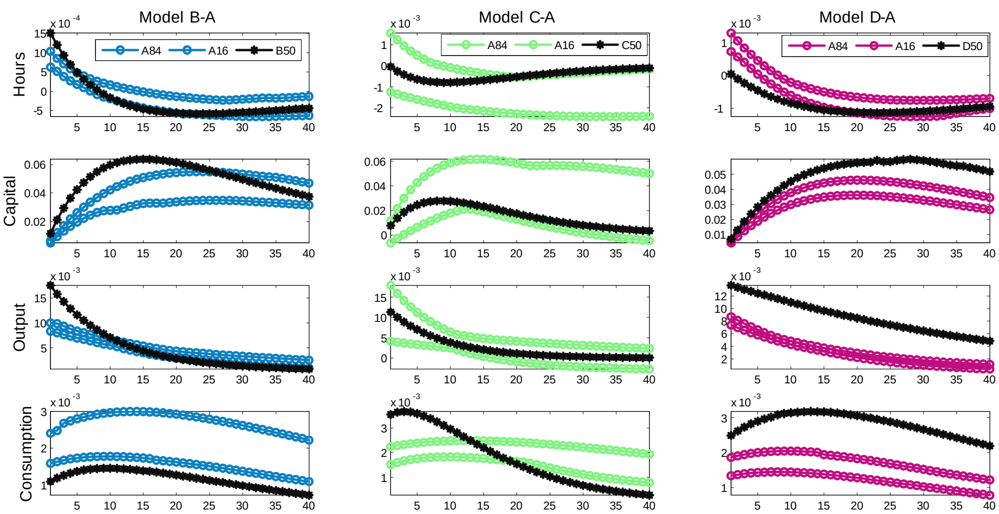

Figure C.5: Impulse responses to technology shocks. Parameter variations explain 15-20 percent of output variance

	Variance attributed to			
	Technology Shocks	Government Shocks	Technology Shocks	Government Shocks
	DGP: Model B			
	True		Estimated: Time invariant	
Y	0.830	0.005	0.827	0.182
C	0.567	0.032	0.298	0.717
N	0.225	0.032	0.099	0.913
K	0.514	0.039	0.412	0.601
	DGP: Model C			
	True		Estimated Time invariant	
Y	0.843	0.005	0.116	0.884
C	0.347	0.024	0.052	0.948
N	0.105	0.034	0.006	0.994
K	0.150	0.027	0.630	0.370
	DGP: Model C			
	True		Estimated Time invariant	
Y	0.826	0.002	0.049	0.966
C	0.325	0.021	0.075	0.943
N	0.063	0.024	0.026	0.985
K	0.215	0.033	0.052	0.963

Table C.5: Long-run variance decomposition. Parameter variations explain 15-20 percent of output variance.

Appendix D : The equations of Gertler and Karadi's model and the prior used

$$
\begin{aligned}
\exp(\varrho_t) &= (\exp(C_t) - h\exp(C_{t-1}))^{-\sigma} - \beta h(\exp(C_{t+1}) - h\exp(C_t))^{-\sigma} && (23)\\
1 &= \beta\exp(R_t)\exp(\Lambda_{t+1}) && (24)\\
\exp(\Lambda_t) &= \frac{\exp(\varrho_t)}{\exp(\varrho_{t-1})} && (25)\\
\chi * \exp(L_t)^{\varphi} &= \exp(\varrho_t)\exp(P_{m,t})(1-\alpha)\frac{\exp(Y_t)}{\exp(L_t)} && (26)\\
\exp(\nu_t) &= (1-\theta)\beta\exp(\Lambda_{t+1})(\exp(R_{k,t+1}) - \exp(R_t)) + \beta\exp(\Lambda_{t+1})\theta\exp(x_{t+1})\exp(\nu_{t+1}) && (27)\\
\exp(\eta_t) &= (1-\theta) + \beta\exp(\Lambda_{t+1})\theta\exp(z_{t+1})\exp(\eta_{t+1}) && (28)\\
\exp(\phi_t) &= \frac{1}{(1-\psi_t)}\frac{\exp(\eta_t)}{\lambda - \exp(\nu_t)} && (29)\\
\exp(z_t) &= (\exp(R_{k,t}) - \exp(R_{t-1}))(1-\psi_{t-1})\exp(\phi_{t-1}) + \exp(R_{t-1}) && (30)\\
\exp(x_t) &= \frac{\exp(\phi_t)(1-\psi_t)}{(\exp(\phi_{t-1})(1-\psi_{t-1}))}\exp(z_t) && (31)\\
\exp(K_t) &= \frac{\exp(\phi_t)\exp(N_t)}{\exp(Q_t)} && (32)\\
\exp(N_t) &= \exp(Ne_t) + \exp(Nn_t) && (33)\\
\exp(Ne_t) &= \theta\exp(z_t)\exp(N_{t-1})\exp(-e_{Ne,t}) + \zeta_t && (34)\\
\exp(Nn_t) &= \omega(1-\psi_{t-1})\exp(Q_t)\exp(\xi_t)\exp(K_{t-1}) && (35)\\
\exp(R_{k,t}) &= (\exp(P_{m,t})\alpha\frac{\exp(Y_{m,t})}{\exp(K_{t-1})} + \exp(\xi_t) * (\exp(Q_t) - \frac{\exp(\delta))}{\exp(Q_{t-1})} && (36)\\
\exp(Y_{m,t}) &= \exp(a_t) * (\exp(\xi_t) * \exp(U_t) * \exp(K_{t-1}))^{\alpha} * \exp(L_t)^{1-\alpha} && (37)\\
\exp(Q_t) &= 1 + 0.5\eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)^2 + \eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)\frac{(In_t + I^s)}{(In_{t-1} + I^s s)}\\
&- \beta\exp(\Lambda_{t+1})\eta_i(\frac{(In_{t+1} + I^s)}{(In_t + I^s} - 1)(\frac{(In_{t+1} + I^s)}{(In_t + I^s)})^2 && (38)\\
\exp(\delta) &= \delta_c + b/(1+\zeta) * \exp(U_t)^{1+\zeta} && (39)\\
\alpha\frac{\exp(Y_m)}{\exp(U_t)} &= \frac{b\exp(U_t)^{\zeta}\exp(\xi_t) * \exp(K_{t-1})}{\exp(P_{m,t})} && (40)\\
In_t &= \exp(I_t) - \exp(\delta) * \exp(\xi_t) * \exp(K_{t-1}) && (41)\\
\exp(K_t) &= \exp(\xi_t) * \exp(K_{t-1}) + In_t && (42)\\
\exp(G_t) &= G^s * \exp(g_t) && (43)\\
\exp(Y_t) &= \exp(C_t) + \exp(G_t) + \exp(I_t) + 0.5\eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)^2(In_t + I^s)\\
&+ \tau\psi\exp(K_t) && (44)\\
\exp(Y_{m,t}) &= \exp(Y_t) * \exp(D_t) && (45)
\end{aligned}
$$

$$
\begin{aligned}
\exp(D_t) &= \gamma * \exp(D_{t-1}) * \exp(infl_{t-1})^{-\gamma_P * \epsilon} \exp(infl_t)^{\epsilon} \\
&+ (1-\gamma)((1-\gamma \exp(infl_{t-1})^{\gamma_P(1-\gamma)} \exp(infl_t)^{\gamma-1})/(1-\gamma))^{-\epsilon/(1-\gamma)} && (46) \\
\exp(X_t) &= 1/\exp(P_{m,t}) && (47) \\
\exp(F_t) &= \exp(Y_t) * \exp(P_{m,t}) \\
&+ \beta\gamma \exp(\Lambda_{t+1}) \exp(infl_{t+1})^{\epsilon} (\exp(infl_t))^{-\epsilon\gamma_P} \exp(F_{t+1}) && (48) \\
\exp(Z_t) &= \exp(Y_t) + \beta\gamma \exp(\Lambda_{t+1}) \exp(infl_{t+1})^{\epsilon-1} \exp(infl_t)^{\gamma_P*(1-\epsilon)} \exp(Z_{t+1}) && (49) \\
\exp(infl_t^*) &= \frac{\epsilon}{\epsilon-1} \frac{\exp(F_t)}{\exp(Z_t)} \exp(infl_t) && (50) \\
(\exp(infl_t))^{1-\epsilon} &= \gamma \exp(infl_{t-1})^{\gamma_P(1-\epsilon)} + (1-\gamma)(\exp(infl_t^*))^{1-\epsilon} && (51) \\
\exp(i_t) &= \exp(R_t) * \exp(infl_{t+1}) && (52) \\
\exp(i_t) &= \exp(i_{t-1})^{\rho_i} (\beta^{-1} \exp(infl_t)^{\kappa_\pi} * (\exp(X_t)/(\epsilon/(\epsilon-1)))^{\kappa_y})^{1-\rho_i} \exp(e_{i,t}) && (53) \\
\psi_t &= \kappa * (R_{k,t+1} - R_t - R_k^s + R^s) + e_{\psi,t} && (54) \\
a_t &= \rho_a * a_{t-1} - \sigma_a * e_{a,t} && (55) \\
\xi_t &= \rho_\xi * \xi_{t-1} - \sigma_\xi * e_{\xi,t} && (56) \\
g_t &= \rho_g * g_{t-1} - e_{g,t} && (57) \\
e_{\psi,t} &= \rho_\psi * e_{\psi,t-1} + e_{\psi,t} && (58)
\end{aligned}
$$

The priors we use in the estimation process are as follows. The parameters common to all models, we use a uniform bounded prior on the persistence and the standard deviation of the capital quality shocks ξ_t, a Beta (0.515, 0.1) prior for the habit parameter h, a Beta (0.4, 0.2) prior for λ, a Beta (0.01, 0.001) prior for ω, and a Beta (0.50, 0.2) prior for θ. The mean of the endogenous variables has a bounded uniform distribution.

For the two additional parameters entering the model with exogenously varying λ_t, we assume that the persistence parameter ρ_λ is truncated normal (0.9, 0.1) and the standard deviation σ_λ is inverted gamma with mean 0.05 and variance 0.02.

For the five additional parameters entering the model with endogenously varying λ, we assume that the persistence parameter ρ_λ is truncated normal(0.9, 0.1) and the standard deviation σ_λ is inverted gamma with mean 0.05 and variance 0.02, $\lambda_u \sim Beta(0.8, 0.05)$; $\phi_1 \sim Beta(0.05, 0.02)$; and $\phi_2 \sim Beta(0.05, 0.02)$.

Additional references

Canova, F. , and M. Paustian (2011). Business cycle measurement with some theory, Journal of Monetary Economics, Elsevier, vol. 58(4), 345-361.

Huang, N. (2014). Weak Inference for DSGE models with time-varying parameters. Boston College, manuscript.

Guerrieri L. and Iacoviello M. (2015) Occbin: A toolkit for solving dynamic models with occasionally binding constraints easily, Journal of Monetary Economics, vol 70(C) 22-38

Koop, G., Pesaran, H., and R. Smith (2013). On the identification of Bayesian DSGE models. Journal of Business and Economic Statistics, 31, 300-314

Lombardo, G., and A. Sutherland (2007). Computing second order accurate solutions for rational expectation models using linear solution methods. Journal of Economic Dynamics and Control, 31, 515-530.

Magnusson, L., and S. Mavroeidis (2014). Identification using stability restrictions. Econometrica 82, 1799-1851.

Rios-Rull J. V., and R. Santaeularia-Llopis, (2010). Redistributive shocks and productivity shocks. Journal of Monetary Economics, 37, 931-948.