ONLINE APPENDIX

Assessing U.S. Aggregate Fluctuations Across Time and Frequencies*

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*The views expressed in this paper are those of the authors and should not be interpreted as those of the Federal Reserve Bank of Richmond, the Federal Reserve System, or the Bank of Finland.

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A Some Background on Wavelets

A.1 Continuous Wavelet Transform

A wavelet $\psi(t)$ is a function of finite length which oscillates around the time axis. The name wavelet (small wave) derives from the admissibility condition, which requires the mother wavelet to be of finite support (i.e., small) and of oscillatory (wavy) behavior. The most commonly used mother wavelet in economic applications - and the one we use in this paper - is the Morlet wavelet defined by $\psi(t) = \pi^{\frac{1}{4}} e^{6it} e^{-\frac{t^2}{2}}$. The continuous wavelet transform of a time series $x(t)$ with respect to a given mother wavelet is:

$$W_x(\tau, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{+\infty} x(t) \overline{\psi} \left(\frac{t-\tau}{s} \right) dt, \tag{1}$$

where $\overline{\psi}$ denotes the complex conjugate of ψ, and τ and s are the two control parameters of the continuous wavelet transform (CWT). The location parameter τ determines the position of the wavelet along the time axis, while the scale parameter s defines how the mother wavelet is stretched. The scale is inversely related to frequency f, with $f \approx 1/s$. A lower (higher) scale means a more (less) compressed wavelet which allows to detect higher (lower) frequencies of the time series $x(t)$.

The ability and flexibility to endogenously change the length of the wavelets is one of the main advantages of the wavelet transform when compared with the most common alternative, the short-time Fourier transform. The wavelet power spectrum (WPS) of $x(t)$ is defined as $(WPS)_x(\tau, s) = |W_x(\tau, s)|^2$. It measures the local variance distribution of the time series $x(t)$ around each time and scale/frequency. The WPS can be averaged over time so that it can be compared to classical spectral methods. In particular, the global wavelet power spectrum (GWPS) can be obtained by integrating the WPS over time: $(GWPS)_x(s) = \int_{-\infty}^{+\infty} W_x(\tau, s) d\tau$.

A.2 Maximal Overlap Discrete Wavelet Transform and Wavelet Multi-Resolution Analysis

Wavelet multiresolution analysis (MRA) allows decomposition of any variable into a trend, a cycle, and a noise component, irrespective of its time series properties. In particular it does not require pre-filtering of the time series to impose stationarity, for instance. This is similar to the traditional time series trend-cycle decomposition approach (Beveridge and Nelson, 1981, and Watson, 1986) or other filtering methods like the Hodrick and Prescott

(1997) or the Baxter and King (1999) band-pass filter. We employ a particular version of the wavelet transform called the Maximal Overlap Discrete Wavelet Transform (MODWT). To perform the MODWT of a given time series we need to apply an appropriate cascade of wavelet filters which is similar to filtering by a set of band-pass filters. This procedure allows to capture fluctuations from different frequency bands.

What distinguishes the wavelet decomposition is that the choice of the scale allows the researcher to hone in on and isolate specific frequency bands that are the objects of interest. While other filtering methods, such as Fourier analysis, also allow a researcher to focus on specific frequencies, a wavelet approach has some key advantages. Traditional decomposition techniques, such as spectral analysis of a time series, tend to impose strong assumptions about the data-generating process. Specifically, they often require data to be stationary or pre-filtered. However, many economic and financial time series are hardly stationary as they exhibit trends and patterns such as structural breaks, volatility clustering, and long memory, which the wavelet approach can handle with ease.

Unlike Fourier analysis, wavelets are defined over a finite window in the time domain, which is automatically and optimally resized according to the frequency of interest and the choice of the scale. Wavelets and standard Fourier analysis are essentially approximations with basis functions, but Fourier basis functions are non-zero almost everywhere, making it harder for them to capture local phenomena. Using a short time window isolates the high-frequency features of a time series, while treating the same signal with a large time window reveals its low-frequency features. By varying the size of the time window, we can therefore capture time-varying and frequency-varying features of the time series at the same time. Wavelets are, thus, very useful when dealing with non-stationary time series, irrespective of whether the non-stationarity comes from the level of the time series (that is, from a long-term trend or jumps) or from higher-order moments (that is, from changes in volatility).

In the paper, we use the MODWT to compute the decomposition. This version is not restricted to a particular sample size: if the data are discrete, the standard wavelet decomposition requires a sample of length 2J for the decomposition to be exact; that is, it imposes a tight restriction on which and how many frequency bands can be considered and might require dropping observations. The MODWT avoids this problem and is also translation-invariant, that is, it is not sensitive to the choice of a starting point for the examined time series. Finally, implementation of the wavelet decomposition requires choice of a specific functional form for the filter that maps the original series into its components.

We follow the literature and choose as a benchmark the Haar filter.

As an example, using the Haar wavelet filter, any variable X_t, regardless of its time series properties, can be decomposed as:

$$X_t = \sum_{j=1}^{J} D_{j,t} + S_{J,t}, \tag{2}$$

where the $D_{j,t}$ are the wavelet coefficients at scale j, and $S_{J,t}$ is the scaling coefficient. These coefficients are given by:

$$D_{j,t} = \frac{1}{2^j}\left(\sum_{i=0}^{2^{j-1}-1} X_{t-i} - \sum_{i=2^{j-1}}^{2^j-1} X_{t-i}\right), \tag{3}$$

$$S_{J,t} = \frac{1}{2^J}\sum_{i=0}^{2^J-1} X_{t-i}. \tag{4}$$

Equations (3) and (4) illustrate how the original series X_t, exclusively defined in the time domain, can be decomposed into different time series components, each defined in the time domain and representing the fluctuation of the original time series in a specific frequency band.

As in the Beveridge and Nelson (1981) time-series decomposition into stochastic trends and transitory components, the wavelet coefficients $D_{j,t}$ can be viewed as components with different levels of calendar-time persistence operating at different frequencies; whereas the scaling coefficient $S_{J,t}$ can be interpreted as the low-frequency trend of the time series under analysis. In particular, when j is small, the j wavelet coefficients represent the higher frequency characteristics of the time series (i.e. its short-term dynamics). As j increases, the j wavelet coefficients represent lower frequencies movements of the series.

A.3 The Wavelet Transform: A Simple Example

The wavelet coefficients resulting from the MODWT with a Haar filter are fairly straightforward to interpret as they are simply differences of moving averages. Consider the case of $J = 1$. A time series X_t is then decomposed into a transitory component D_1 and a persistent scale component S_1 as:

$$X_t = \underbrace{\frac{X_t - X_{t-1}}{2}}_{D_{1,t}} + \underbrace{\frac{X_t + X_{t-1}}{2}}_{S_{1,t}}. \tag{5}$$

When $J = 2$ the decomposition results in two detail components D_1 and D_2 and a scale component D_1:

$$X_t = \underbrace{\frac{X_t - X_{t-1}}{2}}_{D_{1,t}} + \underbrace{\frac{X_t + X_{t-1} - (X_{t-2} + X_{t-3})}{4}}_{D_{2,t}} + \underbrace{\frac{X_t + X_{t-1} + X_{t-2} + X_{t-3}}{4}}_{S_{2,t}}. \tag{6}$$

While the first component D_1 remains unchanged at the now higher scale $J = 2$, the prior persistent component S_1 is divided into an additional transitory component D_2 and a new persistent one S_2. The length K_j of the filter, that is, the number of observations needed to compute the coefficients increases with j: $K_j = 2^j$. Hence, the coarser the scale, the longer the filters.

Intuitively, the lower the frequencies a researcher wants to capture, the wider the time window to be considered. Alternatively, the lower the frequencies targeted, the longer the data sample required. The equations also show that this is a one-sided filter as future values of X_t are not needed to compute the coefficients of the wavelet transform of X_t at time t. This implies that the $D_{j,t}$ and $S_{J,t}$ lag X_t. In other words, they reflect the changes in X_t with some delay. Moreover, since the length of the filters increases with j, so does the delay. Hence, the coarser the scale, the more the wavelet components are lagging behind X_t. Finally, the scale of the decomposition is related to the frequency at which activity in the time series occurs. For example, with annual or quarterly time series, Table A.1 shows the interpretation of the different scales.

Table A.1: Scales and Cycle Length

	Period Length	
Scale j	Annual Data	Quarterly Data
1	2y-4y	2q-4q
2	4y-8y	4q-8q=1y-2y
3	8y-16y	8q-16q=2y-4y
4	16y-32y	16q-32q=4y-8y
5	32y-64y	32q-64q=8y-16y
6	64y-128y	64q-128q=16y-32y
...	>128y	>128q=32y

B Data

We extract aggregate time series from the Haver database, except for per capita real GDP, which we take from the FRED database. Table B1 reports further details on the data and Figure 1 shows the raw data series. GDP growth is computed as the quarter-over-quarter rate. Similarly, our measure of inflation is the quarter-over-quarter growth rate of the PCE price index. We also construct a time series for the spread between the long and the short bond rate, computed as the simple difference.

Table B1: Data

Variable	Mnemonic	Comment
Real GDP per capita	A939RX0Q048SBEA	Seasonally Adjusted
Unemployment	LR@USECON	Seasonally Adjusted, 16 and over
PCE Price Index	JC@USECON	Seasonally Adjusted
Federal Funds Rate	FFED@USECON	Monthly Average of Daily Data

C Wavelet Decompositions

We report the individual wavelet decompositions in Figures 2 - 5 for our variables used in the VAR. The data series enter the VAR as explanatory variables individually or combined in terms of broader frequency bands.[1]

[1] All wavelet series are demeaned in these figures.

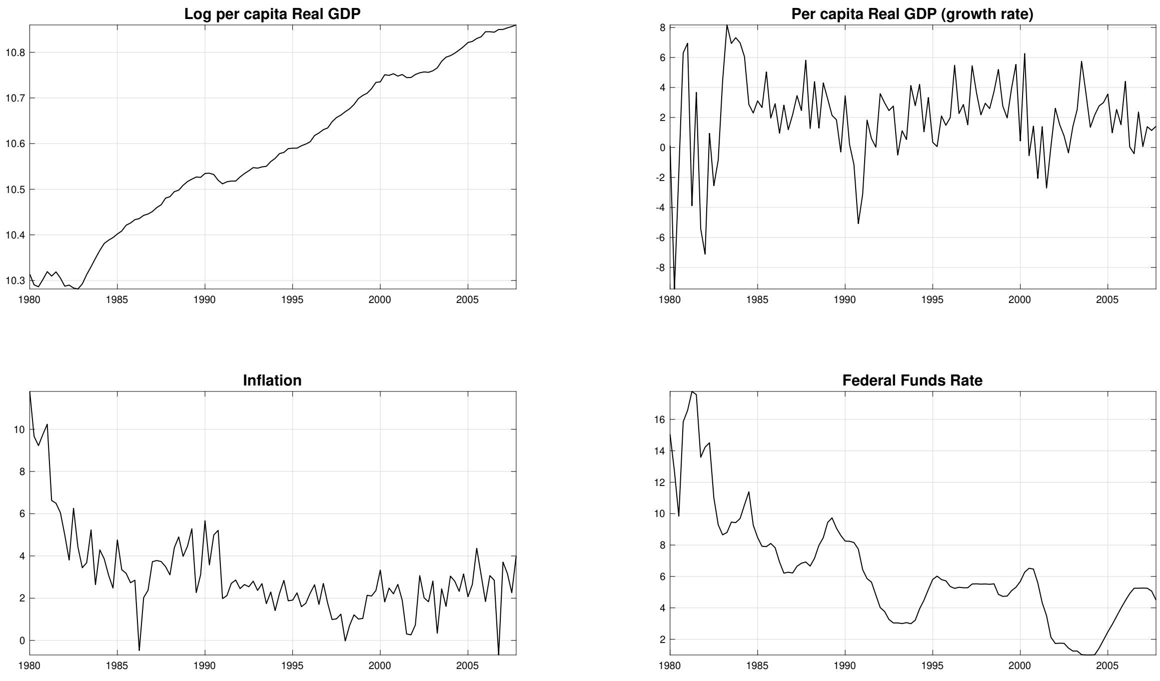

Figure B.1: Data Used in VAR estimation.

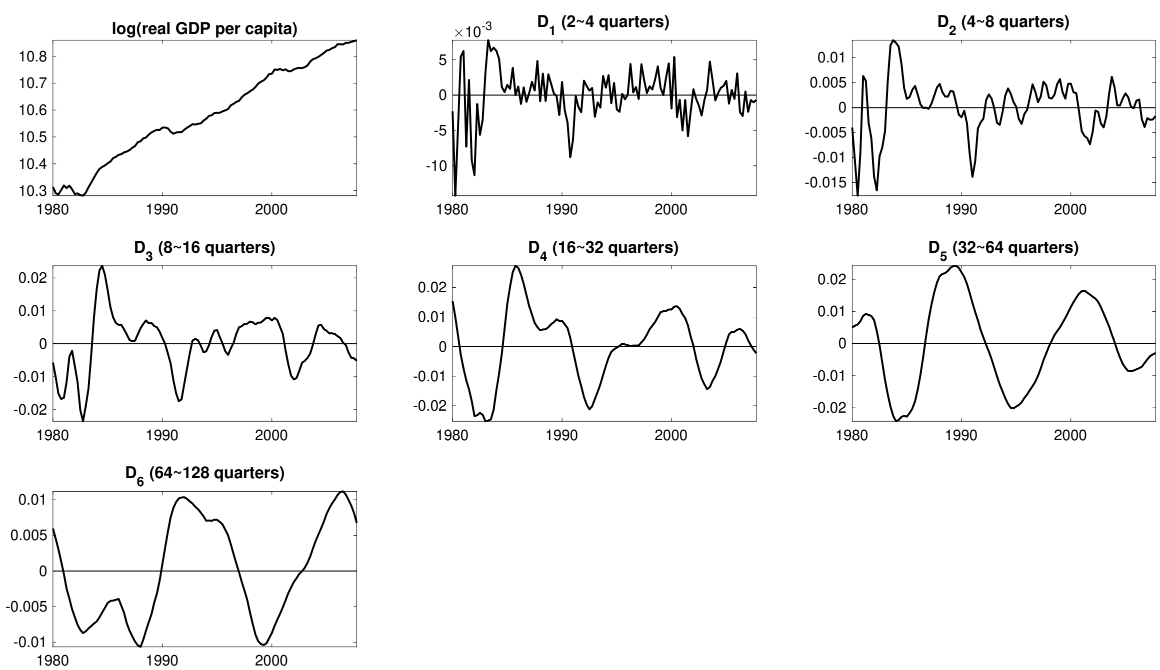

Figure C.1: Log of Real Per Capita GDP and its Wavelet Decomposition.

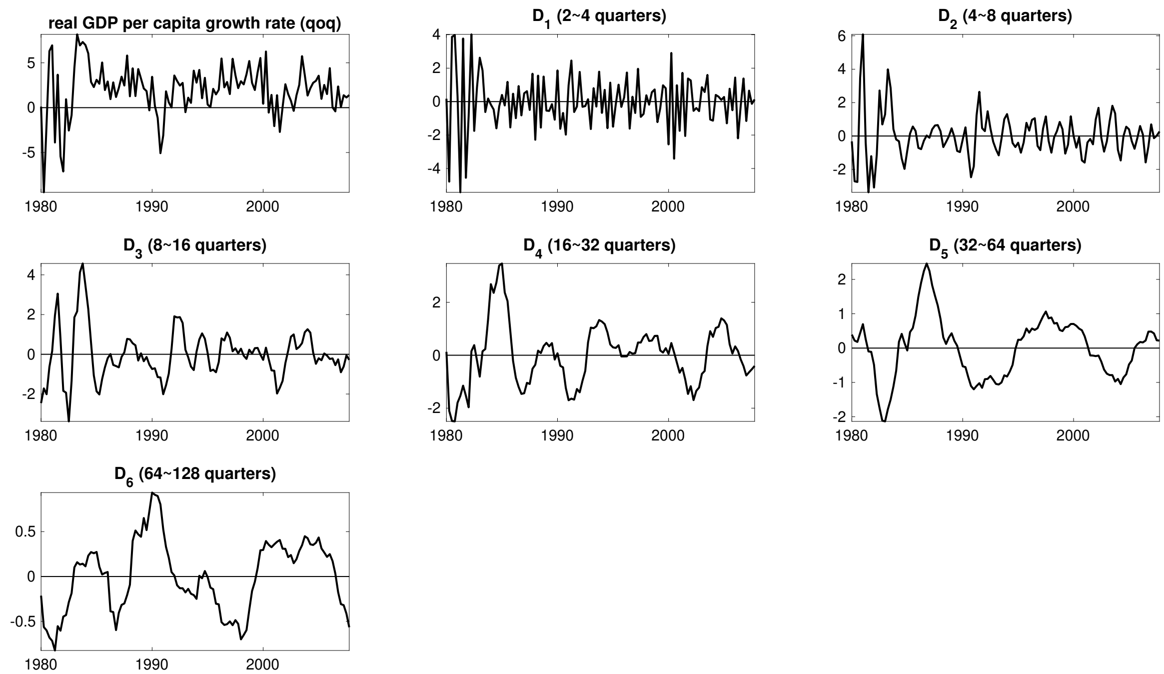

Figure C.2: Growth Rate of Real Per Capita GDP and its Wavelet Decomposition.

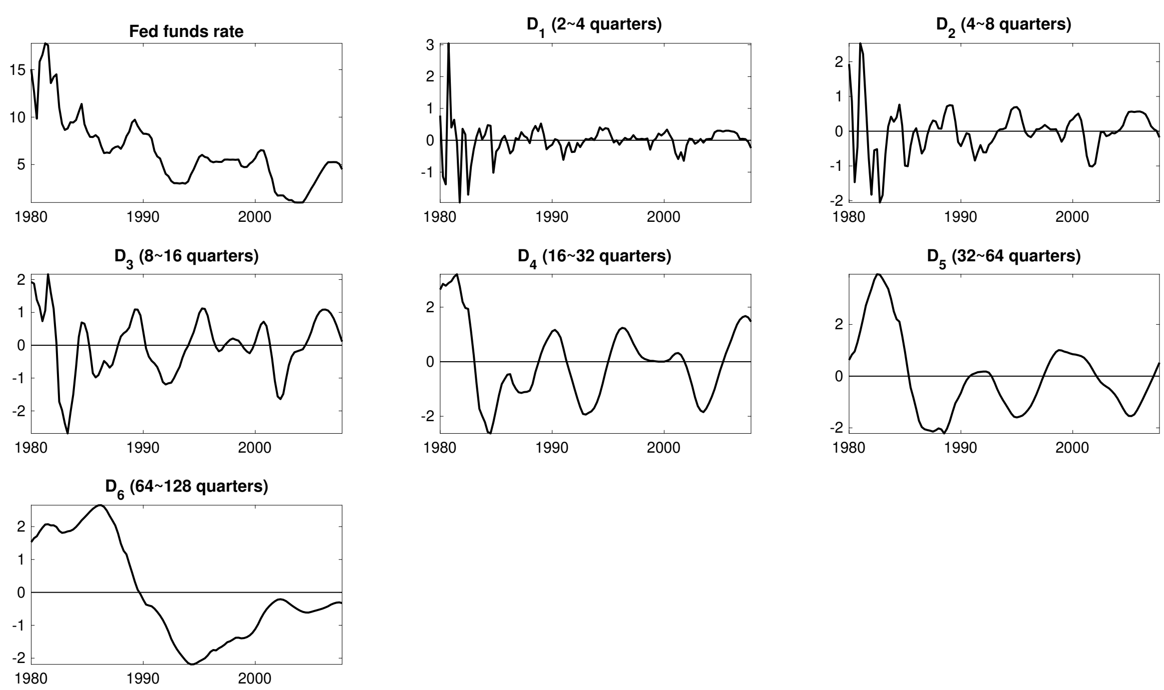

Figure C.3: Federal Funds Rate and its Wavelet Decomposition.

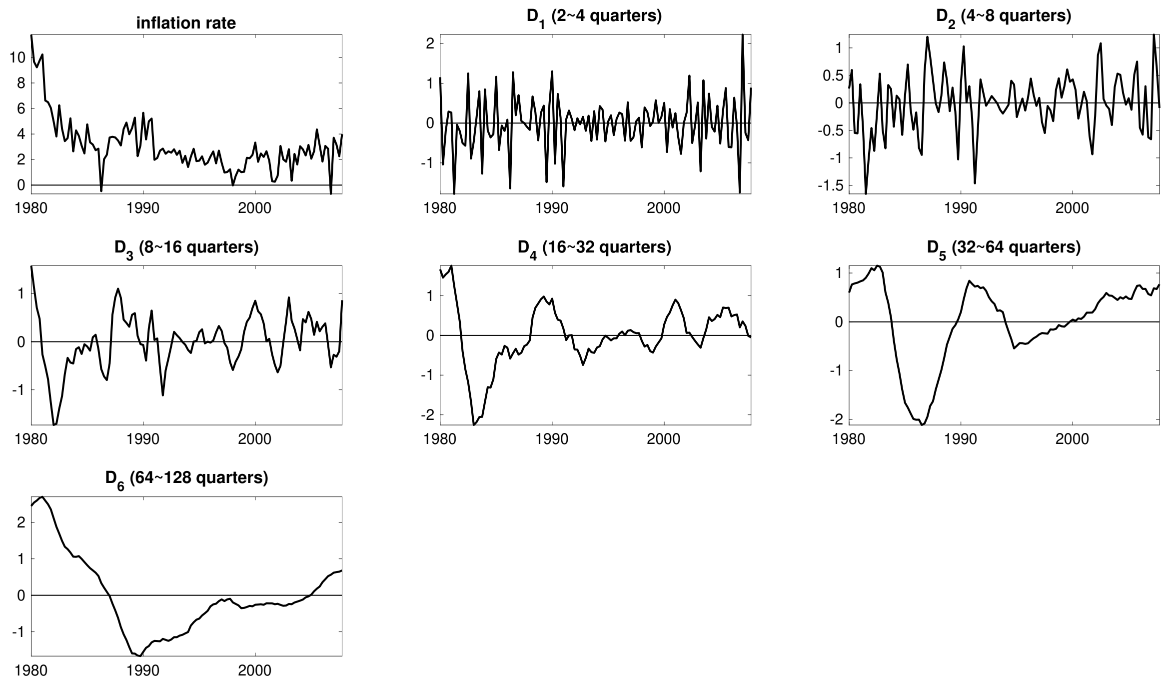

Figure C.4: Inflation and its Wavelet Decomposition.

D DGSE Analysis: Alternative Specifications

Tables D1 - D4 report the variance decompositions for 14 variants of the del Negro et al. (2015) DSGE model. As discussed in the main text, the full model specification captures the individual frequency components quite well. In order to isolate the key model elements we shut down one aspect of the model at a time. The tables show that the specification of the monetary policy rule in terms of the degree of interest-rate smoothing and the aggressiveness in response to inflation movements play a central role. In addition, a time-varying inflation target is central for capturing the medium-term and long-term behavior of inflation. Individual model elements on the real side, such as various adjustment costs to do not play a dominant role.

Table D.1: Real GDP Growth Rate across Models

	real GDP growth rate			
	Short run	Business cycle	Medium run	Long run
baseline model	47	31	18	4
data	55	32	10	3
no price indexation	46	31	19	4
no wage indexation	46	31	18	4
no habit in consumption	61	24	13	2
no investment adj. costs	63	25	11	2
no capital utilization costs	47	31	18	4
flexible prices	53	34	12	2
flexible wages	43	30	22	5
constant inflation target	48	31	18	3
MP only reacts to inflation	55	34	9	1
MP only reacts to inflation (coefficient of 3)	54	36	9	1
no financial frictions	38	28	27	7
higher interest rate smoothing (0.9)	54	31	12	3
no stochastic trend	47	31	18	4

Table D.2: Inflation Rate across Models

	Inflation			
	Short run	Business cycle	Medium run	Long run
baseline model	17	18	23	43
data	16	18	26	39
no price indexation	16	13	23	48
no wage indexation	17	18	23	42
no habit in consumption	16	17	23	44
no investment adj. costs	17	18	23	42
no capital utilization costs	17	18	23	42
flexible prices	41	25	16	17
flexible wages	11	18	32	39
constant inflation target	32	28	22	18
MP only reacts to inflation	21	20	21	38
MP only reacts to inflation (coefficient of 3)	37	30	18	14
no financial frictions	18	19	26	38
higher interest rate smoothing (0.9)	11	14	28	47
no stochastic trend	17	19	26	39

Table D.3: Interest Rate across Models

	Federal Funds Rate			
	Short run	Business cycle	Medium run	Long run
baseline model	8	24	35	32
data	5	18	25	53
no price indexation	8	25	36	31
no wage indexation	8	25	36	31
no habit in consumption	6	18	36	40
no investment adj. costs	8	23	38	31
no capital utilization costs	8	23	35	33
flexible prices	9	24	35	32
flexible wages	5	22	43	29
constant inflation target	11	31	41	17
MP only reacts to inflation	14	28	26	32
MP only reacts to inflation (coefficient of 3)	15	39	32	13
no financial frictions	6	22	45	27
higher interest rate smoothing (0.9)	4	15	35	46
no stochastic trend	8	24	38	30

Table D.4: Log per capita GDP across Models

	log (real GDP per capita)			
	Short run	Business cycle	Medium run	Long run
baseline model	0	2	12	87
data	0	1	8	91
no price indexation	0	2	12	86
no wage indexation	0	2	12	86
no habit in consumption	0	2	11	88
no investment adj. costs	0	2	12	85
no capital utilization costs	0	2	12	86
flexible prices	0	2	10	89
flexible wages	0	2	14	83
constant inflation target	0	2	11	87
MP only reacts to inflation	0	1	9	89
MP only reacts to inflation (coefficient of 3)	0	2	9	89
no financial frictions	0	2	16	81
higher interest rate smoothing (0.9)	0	1	11	88
no stochastic trend	0	2	11	87

References

[1] Baxter, Marianne, and Robert G. King (1999): "Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series". *The Review of Economics and Statistics*, 81(4), pp. 575-593.

[2] Beveridge, Stephen, and Charles R. Nelson (1981): "A New Approach to Decomposition of Economic Time Series into Permanent and Transitory Components with Particular Attention to Measurement of the 'Business Cycle' ". *Journal of Monetary Economics*, 7(2), pp. 151-174.

[3] Del Negro, Marco, Marc P. Giannoni, and Frank Schorfheide (2015): "Inflation in the Great Recession and New Keynesian Models". *American Economic Journal: Macroeconomics*, 7(1), pp. 168-96.

[4] Hodrick, Robert, and Edward C. Prescott (1997): "Postwar U.S. Business Cycles: An Empirical Investigation". *Journal of Money, Credit, and Banking*, 29(1), pp. 1-16.

[5] Watson, Mark W. (1986): "Univariate Detrending Methods with Stochastic Trends". *Journal of Monetary Economics*, 18, pp. 49-75.