Indeterminacy and Imperfect Information*

Thomas A. Lubik
Federal Reserve Bank of Richmond[†]

Christian Matthes
Indiana University[‡]

Elmar Mertens
Deutsche Bundesbank[§]

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Abstract

We study equilibrium determination in an environment where two types of agents have different information sets: Fully informed agents observe histories of all exogenous and endogenous variables. Less informed agents observe only a strict subset of the full information set and need to solve a dynamic signal extraction problem to gather information about the variables they do not directly observe. Both types of agents know the structure of the model and form expectations rationally. In this environment, we identify a new channel that generates equilibrium indeterminacy: Optimal information processing of the less informed agent introduces stable dynamics into the equation system that lead to self-fulling expectations. For parameter values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium is indeterminate. We illustrate our framework with monetary policy models where an imperfectly informed central bank follows an interest rate rule.

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[†]Research Department, P.O. Box 27622, Richmond, VA 23261. Email: thomas.lubik@rich.frb.org.
[‡]Department of Economics, Wylie Hall 202, Bloomington, IN 47405. Email: matthesc@iu.edu.
[§]Research Centre, Wilhelm-Epstein-Str. 14, 60431 Frankfurt, Germany. Email: elmar.mertens@bundesbank.de.

1 Introduction

Asymmetric information is a pervasive feature of economic environments. Even when agents are fully rational, their expectation formation and decision-making process are constrained by the fact that information may be imperfectly distributed in the economy. Asymmetric information is also a central issue for the conduct of monetary policy as policymakers regularly face uncertainty about the true state of the economy, for instance, because they receive data in real time that are subject to measurement error. In environments where information is perfect and symmetrically shared, the literature has shown that policy rules can cause indeterminacy, unless they respond to economic outcomes with sufficient strength. We study equilibrium determinacy in an asymmetric information setting, where policy is conducted based on policymakers' optimal estimates of economic outcomes rather than their true values.

We consider economic environments with two types of agents, one who has full information about the state of economy while the other agent has limited information. We think of the two agents as a representative private-sector agent and a less informed policymaker. While the representative agent observes aggregate outcomes without error, the imperfect information of the policymaker can, for example, take the form of aggregate data subject to measurement error.[1]

A key assumption of our modelling framework is that both types of agents, the policymaker and the private sector, employ rational expectations, but based on different information sets. Private-sector behavior is characterized by a set of linear, expectational difference equations. On the other hand, the policymaker's behavior is characterized by an instrument rule, which responds to the policymaker's estimates of economic conditions. Specifically, we consider the conduct of monetary policy with a Taylor-type interest-rate rule that responds to the policymaker's projection of current inflation rather than its actual value. Formally, we consider linear, stochastic equilibria with time-invariant decision rules and Gaussian shocks. In this case, the rational inference efforts of the policymaker are represented by a dynamic signal extraction problem as captured by the Kalman filter. The interaction of the two expectation formation processes is the source of the new mechanism underlying equilibrium multiplicity in our environment.

The central result of our paper is that indeterminacy is widespread for a broad class of linear imperfect-information models that have unique equilibria under full information. In our imperfect information setup, optimal information processing of the less informed agent introduces stable dynamics into the equation system that can transmit self-fulling belief shocks. The interaction of the two expectation processes generates an endogenous feedback mechanism similar to strategic complementarities or the application of ad-hoc behavior in the standard indeterminacy literature.

[1] Such a dichotomy is well-established in the learning literature. Where our work differs is that both agents have rational expectations and know the structure of the economy, although not necessarily realizations of its state.

When there is indeterminacy in the perfect-information case, there are no restrictions on the scale and direction of effects caused by belief shocks. In contrast, the potential effects of belief shocks are tightly bounded in our imperfect information environment. The bounds arise from the required consistency of expectations of the public and the policymaker and the assumption that we consider only environments that have a unique equilibrium under full information. While the rationality of expectations under both information sets places non-trivial restrictions on outcomes, they are not sufficient to rule out multiple equilibria. Moreover, the interplay of expectations based on different information sets results in equilibrium outcomes that are not certainty equivalent even though we only consider environments that are linear.

We illustrate key insights from our framework in three models of inflation determination, where monetary policy follows a Taylor-type interest-rate rule. The rule satisfies the Taylor principle, that is, it responds more than one-for-one to the policymaker's reading of current inflation, which guarantees determinacy under full information. Under imperfect information, however, the policymaker does not observe all equilibrium outcomes, and the policy rule responds only to optimal projections of current endogenous outcomes. The sensitivity of the policy rate to movements in the endogenous variables thus depends on the sensitivity of the policymaker's projection to the incoming signal. The more the central bank is successful at stabilizing inflation, for example by reacting more aggressively to projected inflation, the noisier will be the signal and the policymaker's projection will barely respond. By the same logic, non-fundamental shocks cannot become an arbitrarily large driver of inflation. Otherwise, the central bank's signal would become highly informative and the policy rate would respond with sufficient strength to actual inflation to re-establish determinacy.

Our paper touches upon three strands in the literature. First, we contribute to the burgeoning literature on imperfect information in macroeconomic models that focuses on the implications of dispersed information among different members of the public and the resulting effects on their strategic interactions and the informational value of prices. Key contributions by Nimark (2008a, 2008b, 2014), Angeletos and La'O (2013), and Acharya, Benhabib, and Huo (2021) demonstrate that imperfect information has important implications for the amplification and propagation of economic shocks. Papers in this literature often feature non-nested information sets, which leads to an infinite number of state variables. Our assumptions on the information structure allow us to derive a finite state representation of the solution to our models.

In Angeletos and La'O (2013), aggregate fluctuations are driven by non-fundamental shocks, which are modelled explicitly as exogenous shocks to agents' beliefs, and the equilibrium is unique and always exists. In contrast, the equilibrium in Benhabib, Wang, and Wen (2015) is unique without non-fundamental shocks. Their sentiment shocks arise endogenously and their existence and statistical properties depend on the model's primitives. Acharya, Benhabib, and Huo (2021) extend the notion of this sentiment equilibrium to allow for persistent sentiment processes. Our paper is similar to this work in the sense that the sunspot shocks arise endogenously, and that by adding imperfect information the unique equilibrium property fails to hold. We differ in that we explore forward-looking behavior and consider hierarchy of information structure instead of dispersed information. In that respect, our framework is closer to Rondina and Walker (2021). More recently, Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017) have studied information externalities in the private sector, which lead to uncertainty traps, namely self-reinforcing periods of high uncertainty and low activity. Similarly, Angeletos, Iovino, and La'O (2020) and Kozlowski, Veldkamp, and Venkateswaran (2020) model endogenous learning in the private sector which can be seen as complementary to our setup of endogenous learning by policymakers.[2]

Second, our research makes a contribution to the literature on indeterminacy in linear rational expectations models by expanding the set of plausible economic mechanisms that can lead to multiple equilibria. A key element of the indeterminacy literature is the presence of a mechanism that validates self-fulfilling expectations. These could arise from what is often termed strategic complementarities, such as increasing returns to scale in production that are not internalized, as in the seminal contributions of Benhabib and Farmer (1994), Farmer and Guo (1994), and Schmitt-Grohe (1997). An alternative mechanism is the interplay between economic agents' forward-looking behavior and the reaction function of a policymaker, which Clarida, Gali, and Gertler (2000) and Lubik and Schorfheide (2004) show to be a key feature of macroeconomic fluctuations.[3]

In contrast, our framework does not rely on these previously identified sources of indeterminacy but rather on the interaction of different expectation formation processes under asymmetric information. This also sets our framework apart from the broader imperfect information literature, which is largely concerned with the strategic interaction between agents in the private sector. Although our framework utilizes the root-counting formalism of the indeterminacy literature, where we build on the contributions of Lubik and Schorfheide (2003, 2004) and Farmer, Khramov, and Nicolò (2015), the mechanism to get there is novel. Critically, while equilibria are described by solutions to linear difference systems, the roots of these dynamic systems depend on endogenous Kalman gains and are not invariant to the equilibrium outcomes. We show that the set of multiple equilibria, despite the pervasiveness of indeterminacy, is tightly circumscribed by internal consistency requirements for the interaction between the two expectation processes. Our paper thereby puts some caveats on the notion that sunspot shocks are unrestricted in their effects on macroeconomic outcomes.

[2] We focus on the case of an imperfectly informed policymaker while treating the private sector as fully informed. However, imperfect information of the private sector is an additional concern to be studied. For example, Nakamura and Steinsson (2018) and Jarocinski and Karadi (2020) point to an information effect from policy changes on private-sector behavior although the quantitative relevance of this effect has recently been questioned by Bu, Rogers, and Wu (2021) and Bauer and Swanson (2020). However, the key ingredient for our indeterminacy result is the policymaker's inability to respond directly to self-fulfilling belief shocks, which could otherwise be used to rule out indeterminate equilibria.

[3] Ascari, Bonomolo, and Lopes (2019) also point to sunspot-driven equilibria as a key source of fluctuations during the high-inflation era of the 1970s.

Third, the examples in our paper speak to the monetary policy literature concerned with the effects of interest-rate rules on determinacy. A well-known result from this literature is the Taylor principle, which requires that interest rate rules respond to endogenous variables with sufficient strength, to avoid multiple equilibria and ensure determinacy. Clarida, Gali, and Gertler (2000) and Lubik and Schorfheide (2004) have pointed to a neglect of the Taylor rule as a possible factor behind the Great Inflation. However, their evidence is based on a full-information perspective that does not account for the uncertainties faced by the Federal Reserve in assessing the state of the economy in real time, as discussed by Orphanides (2001). Our results point to the critical role played by the central bank's (in)ability to observe the output gap (and other policy-relevant variables) when seeking to design rules that conform to the Taylor principle.[4]

Similar to our framework, Orphanides (2003) models the economic consequences of an imperfectly informed central bank that responds to estimates of economic conditions generated by optimal signal extraction efforts. But in a fundamental difference to our framework, his model is purely backward-looking so that the issue of indeterminacy does not arise. Our paper also relates to Svensson and Woodford (2004) and Aoki (2006) who derive conditions for optimal policy when the policymaker is less informed than the public in forward-looking linear rational expectations models, but take determinacy as given.[5]

We proceed as follows. In the next section, we introduce and motivate our framework by means of simple examples drawn from the monetary policy literature, for which we can derive analytical results. Section 3 provides a general description of the framework. We present a general linear rational expectations framework with asymmetric information sets and use results from general linear systems theory to derive properties of the resulting equilibria. We present three quantitative examples in Section 4. Section 5 concludes and discusses further extensions of our framework. An online appendix provides additional results and detailed derivations.

2 Three Motivating Examples

We introduce our framework with three simple examples. The first example uses the canonical New Keynesian (NK) model, where potential output is unobserved by the central bank. Under this

[4] As documented by Orphanides and van Norden (2002) and the ensuing literature, challenges in output gap filtering are typically compounded by real-time uncertainty about the actual levels of inflation and output. Our quantitative analysis also considers the joint effects of real-time uncertainty about inflation, output and the output gap.

[5] Applications of the framework by Svensson and Woodford (2004) to various economic issues are Carboni and Ellison (2011), Dotsey and Hornstein (2003), and Nimark (2008b). Evans and Honkapohja (2001) and Orphanides and Williams (2006, 2007) revisit the question of policymaking under imperfect information in an environment with learning. In a similar vein, Lubik and Matthes (2016) explore indeterminacy in a setting where the central bank employs constant-gain learning, however, without imposing that the policymaker employs rational expectations. Faust and Svensson (2002) and Mertens (2016) study the implications for optimal policy of the opposite informational asymmetry, where the public does not perfectly share the policymaker's information set.

specification, rational expectations equilibria are indeterminate even for policy rules that imply uniqueness under full information. The second example continues with the NK specification, but assumes that actual output is unobserved. A similar indeterminacy problem arises, which we use to illustrate that the information set of the central bank matters for equilibrium determination. In the third example, we switch to a simple Fisher-equation model for analytical expediency. We introduce measurement error into this otherwise standard information setting and show that indeterminacy again arises. As we progress through the examples, each example gives the central bank less information than in the previous example. Consequently, the imperfect information outcomes offer richer insights. In all examples we use exogenous $i.i.d.$ shocks, but the second and third examples feature endogenous persistence due to imperfect information. Details of the derivations are relegated to the Appendix.

The first two examples are based on the canonical three-equation NK model, which we replicate below for reference. In our framework there are two types of expectations: Those held by the less-informed central bank and those held by the public. The central bank's expectations of inflation are defined as $\pi_{t|t} \equiv E\left(\pi_t \mid Z^t\right)$, where $Z^t = \{Z_t, Z_{t-1}, ...\}$ denotes the entire history of signals the central bank receives (a similar definition holds for other variables and one-step ahead expectations). We denote private sector expectations with the operator E_t. The model consists of an Euler equation for the output gap x_t:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \overline{r}_t\right),$$

where i_t is the nominal interest rate, π_t is inflation, and $\overline{r}_t$ is the natural real interest rate. E_t denotes the standard rational expectations operator conditional on the full information set. The NK Phillips curve is:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t,$$

where u_t is a cost-push shock. A key element in our framework is that the central bank sets policy based on optimal projections that conform with rational expectations under limited information as in Svensson and Woodford (2004). Accordingly, the third equation is a Taylor-type monetary policy rule, but responding to central bank projections of its target variables:

$$i_t = \phi_\pi \pi_{t|t} + \phi_x x_{t|t}\,. \tag{1}$$

The presence of central bank projections in the policy rule connects projected and actual outcomes under the two information sets and leads to a new channel through which equilibria occur. We discuss the specific equilibrium concept in section 3, but note here that indeterminacy is widespread in our framework.

The model is closed by specifying a law of motion for the natural interest rate:

$$\overline{r}_t = \sigma \left(E_t \overline{y}_{t+1} - \overline{y}_t \right),$$

where $\overline{y}_t$ is (exogenous) potential output so that actual output $y_t = \overline{y}_t + x_t$. Finally, we specify the two shocks as $u_t \sim_{iid} N\left(0; \sigma_u^2\right)$ and $\overline{y}_t \sim_{iid} N\left(0; \sigma_{\overline{y}}^2\right)$.

Under full information, the central bank observes potential output and all shocks directly, and its projections for current outcomes collapse to their actual values, $\pi_t = \pi_{t|t}$ and $x_t = x_{t|t}$. Assuming that the policy coefficients are such that they imply a unique full-information equilibrium (Bullard and Mitra 2002), the rational expectations solution is:

$$\pi_t = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_t - \frac{\kappa\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_t, \tag{2}$$

$$x_t = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_t - \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_t, \tag{3}$$

$$y_t = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_t + \frac{\phi_x + \kappa\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_t. \tag{4}$$

2.1 A New Keynesian Model with Observed Output and Inflation

In our first example, we assume that the central bank observes output and inflation, but not the output gap. This is arguably a plausible case since potential output is a fundamentally unobservable variable, yet it is at the core of modern monetary policy frameworks. We show that this seemingly minimal deviation from the full-information assumption has stark implications. For simplicity, we assume that potential output is iid, so that $\overline{r}_t = -\sigma\, \overline{y}_t$.

The model equations can be conditioned down onto the central bank's limited information set, and we obtain a system of equations that is isomorphic to the full-information solution in (2) – (4) except for using the (yet to be determined) central bank projections in lieu of actual values:

$$\pi_{t|t} = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\kappa\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{5}$$

$$x_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{6}$$

$$y_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} + \frac{\phi_x + \kappa\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}. \tag{7}$$

We refer to these equations as *projection condition* and show later in Section 3 that the same principle carries over to our general framework: Stationarity of a linear and time-invariant equilibrium requires that the full-information mapping from backward- to forward-looking variables has to hold also for their projected values under imperfect information.[6]

With inflation and output directly observed by the policymaker we have $\pi_{t|t} = \pi_t$ and $y_{t|t} = y_t$. However, the central bank still needs to form projections of the output gap to evaluate its policy rule $i_t = \phi_\pi \pi_t + \phi_x x_{t|t}$. Since the central bank observes as many outcome variables as there are fundamental shocks, the projection condition of equations (5) – (7) pins down the equilibrium responses of the policymaker's output gap projection to observables:

$$x_{t|t} = -\frac{\sigma}{\phi_x} y_t - \frac{\phi_\pi}{\phi_x} \pi_t \tag{8}$$

This equation represents an equilibrium restriction on the solution to the central bank's signal extraction problem. After establishing that there is indeterminacy in this model, we discuss the consequences of these restrictions for the roles played by fundamental and non-fundamental shocks in the remainder of this section.

Substitution of the output gap projection in (8), into the limited-information policy rule (1), together with $\pi_t = \pi_{t|t}$, results in an "effective" Taylor rule of the following form:

$$i_t = \bar{\phi}_\pi \pi_t + \bar{\phi}_y y_t \,, \tag{9}$$

Moreover, as shown in the appendix, we obtain the coefficients $\bar{\phi}_\pi = 0$ and $\bar{\phi}_x = -\sigma$ for all parameter values of the underlying limited-information Taylor rule (1). Determinacy with an outcome-based rule as in (9) requires that $\kappa(\bar{\phi}_\pi - 1) + (1-\beta)\bar{\phi}_y > 0$ (Bullard and Mitra 2002).[7] However, with $\bar{\phi}_\pi = 0$ and $\bar{\phi}_x = -\sigma$ this determinacy condition is never satisfied since $-\kappa - (1-\beta)\sigma < 0$. Consequently, the rational expectations equilibrium, if it exists, is indeterminate, irrespective of the value of the underlying policy coefficients and the strength of the central bank's anti-inflationary policy stance. Incidentally, the effective Taylor rule in this example reduces to an attempt of letting the (nominal) policy rate track the real rate $i_t = -\sigma y_t = \sigma(E_t y_{t+1} - y_t) = r_t$. This is similar to a policy of pegging the policy rate (in this case to the time-varying real rate), which is known to lead to indeterminacy (Sargent and Wallace 1975).

At this point, we have established that any equilibrium is indeterminate in this setting since the equation system fails the root counting condition of Blanchard and Kahn (1980). In the next step, we derive a full characterization of the rational expectations solution, which includes sunspot shocks under indeterminacy. We first provide a general characterization of the reduced-form dynamics and then detail how the projection condition limits the set of possible equilibria (but without uniquely

[6] In a setup similar to ours, Svensson and Woodford (2004) also embed a set of equations akin to the projection condition (and based on a mere appeal to certainty equivalence). In contrast to our paper, they restrict the number of state variables in a way that disregards a role for belief shocks, and assume equilibrium uniqueness.

[7] The analysis of Bullard and Mitra (2002) is concerned with rules responding to the output gap, x_t, instead of output, y_t. However, since $y_t = x_t + \bar{y}_t$ and $\bar{y}_t$ exogenous, their determinacy condition also applies in our case.

determining a single equilibrium).[8]

We can solve the Phillips curve forward, which results in a solution for the inflation process $\pi_t = \kappa x_t + u_t$. The Euler-equation then implies that $E_t x_{t+1} = 0$, which leaves x_t unrestricted. Following Farmer, Khramov, and Nicolò (2015) we can therefore specify a generic solution $x_t = \gamma_u u_t + \gamma_{\overline{y}} \overline{y}_t + \gamma_b b_t$, where $b_t \sim_{iid} N(0,1)$ is a normalized sunspot or belief shock, while γ_u, $\gamma_{\overline{y}}$, and γ_b are the respective loadings on the shock processes that are yet to be determined.

A key insight from our framework is that any proposed equilibrium has to be consistent with central bank projections (and vice versa), which leads to the projection condition. This condition recognizes that the fully informed private sector is aware of the policymaker's signal-extraction problem under limited information. The central bank's projection is required to be optimal given the environment, including the private sector's expectations of the central bank's problem. We therefore posit that the private sector would only coordinate on indeterminate equilibria that are consistent with optimal central bank projections, which therefore validate these beliefs. Conceptually, this results in a non-linear fixed point problem that makes analytical solutions hard to obtain in general, except for some of the special cases considered in this section.

The projection condition poses additional restrictions on the shock loadings γ_u, $\gamma_{\overline{y}}$, and γ_b. Specifically, we can derive a restriction that limits the covariance between outcomes and signals observed by the policymaker, which in turn places a bound on the economy's response to a belief shock. We can rewrite the imperfect-information solution in terms of deviations from the full-information case, that are captured by a univariate discrepancy measure, δ_t:

$$x_t = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_t - \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_t + \delta_t,$$

$$\text{where} \quad \delta_t = \left(\gamma_u + \frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi}\right) u_t + \left(\gamma_{\overline{y}} + \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi}\right) \overline{y}_t + \gamma_b b_t.$$

The projection condition then requires that the central bank's projection of the discrepancy δ_t has to be zero, $\delta_{t|t} = 0$. Since the full-information solution corresponds to $\delta_t = 0$, it trivially fulfills the projection condition.

To satisfy the projection condition, δ_t does not have to be zero, but merely uncorrelated with π_t and y_t at all times and at all lags. Since δ_t, π_t and y_t are iid in this specific example, this implies that the following covariance conditions are necessary and sufficient for the projection condition to hold, namely $cov(\delta_t, \pi_t) = 0$ and $cov(\delta_t, y_t) = 0$. These are two non-linear equations in three unknowns, which restrict the set of possible solutions under indeterminacy, without uniquely identifying a single equilibrium.

[8] In our specific $i.i.d.$ example, the equation system has a singular transition matrix. Singularity implies that indeterminacy only leads to amplification without additional propagation; that is, belief shocks affect variables on impact, but without persistence. We solve a version with persistent driving processes numerically in section 4.

Crucially, the projection condition restricts the scale of the belief-shock loading γ_b, which is a property absent from indeterminate equilibria under full information. As shown in Appendix A.1, we have:

$$\gamma_b^2 < \frac{1}{4 \cdot a}, \qquad \text{with} \quad a = \left[\frac{\kappa^2}{\sigma_u^2} + \frac{1}{\sigma_{\bar{y}}^2}\right] \cdot \left(\frac{\sigma + \kappa\phi_\pi + \phi_x}{\phi_x}\right)^2 . \tag{10}$$

This belief-shock bound gets wider the larger the variances of fundamental shocks, σ_u^2 and $\sigma_{\bar{y}}^2$ are, and the less aggressive policy rule is (i.e. the smaller the coefficients ϕ_π and ϕ_x are).

Why does a seemingly small deviation from full-information rational expectations deliver such a stark outcome of indeterminacy? Suppose a sunspot shock lets private agents believe that inflation is higher than fundamentals warrant. Under full information, and as long as the policy rule satisfies the Taylor principle, policy rates would be raised so aggressively that these beliefs cannot be validated. Under limited information, however, policy can only respond to the projected output gap. As a result of using optimal projections, the policy reaction to actual outcomes becomes sufficiently attenuated that these beliefs can take hold. The reason is that higher inflation is attributed, at least in part, to a cost-push shock, which lowers the central bank's output gap projection. Due to these offsetting effects from observing high inflation, the policy rate does not move sufficiently, which in turn validates the self-fulfilling sunspot belief. The projection condition restricts the extent of how much this indeterminate equilibrium differs from its full-information counterpart, but not the fact that it does.

A key insight of our framework is that under indeterminacy non-fundamental sunspot shocks come into play. In our specific example setting, the central bank observes as many variables as there are fundamentals shocks in the economy. One might be tempted to conclude that the central bank could perfectly infer the underlying shocks and thus the output gap. However, this conclusion would have to rest on the added assumption that the equilibrium was uniquely determined, and that only fundamental shocks drive fluctuations. Generally, in case of indeterminacy, sunspot shocks become also relevant, so that the total number of shocks hitting the economy is larger than the number of variables observed by the central bank. This leads to a signal extraction problem that ultimately attenuates policy responses to fall below what is needed to assure determinacy.[9]

In Appendix A.2 we study an extension of the simple NK model that adds interest-rate smoothing to the policy rule. Crucially, the model generates generic indeterminacy, and a belief-shock bound isomorphic to (10). Moreover, in case of interest-rate smoothing, the effects of belief shocks propagate to future outcomes via their effects on interest rates.

[9]This mechanism is an application of invertibility in rational expectations frameworks. Having as many shocks as observables is, at best, only a necessary condition for such invertibility, whereby a set of underlying shocks can be perfectly inferred from a set of measurement variables; see, for example Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007) or Rondina and Walker (2021).

2.2 A New Keynesian Model with Observed Inflation Only

In the second example, we continue with the simple NK model, but change the information set. We now assume that the central bank only observes inflation. For analytical tractability, we focus on the case with cost-push shocks only. In the $i.i.d.$ case, the full-information solution is as presented above with potential output, and thus the real rate, constant at zero. In our imperfect information setup, the solution for the equation system when conditioned down to the central bank's information set then implies $\pi_{t|t} = \frac{\phi_x+\sigma}{\phi_\pi\kappa+\phi_x+\sigma}u_{t|t}$ and $x_{t|t} = -\frac{\phi_\pi}{\phi_\pi\kappa+\phi_x+\sigma}u_{t|t}$, which is also the projection condition discussed before.

When the central bank observes inflation without error, $\pi_{t|t} = \pi_t$. The projection condition then implies that $\pi_{t|t-1} = 0$ and $x_{t|t-1} = 0$ since the shocks are iid and so is the solution to the conditioned-down system. We can then derive $x_{t|t} = -\frac{\phi_\pi}{\phi_x+\sigma}\pi_t$, which is the Kalman-updating equation, as in the previous example, and the coefficient on observable inflation is the Kalman gain. Notably, it is pinned down by fundamental parameters only, independently of the shock variances as would typically be the case. With the Kalman gain determined as above, the model is isomorphic to the full-information case with a policy rule of the form $i_t = \bar{\phi}_\pi\ \pi_t$, where the output gap projection is substituted out. The effective policy coefficient $\bar{\phi}_\pi = \phi_\pi - \frac{\phi_x\phi_\pi}{\phi_x+\sigma} = \phi_\pi\frac{\sigma}{\phi_x+\sigma} < \phi_\pi$. If $|\bar{\phi}_\pi| > 1$ the solution is determinate and there is a single shock, u_t, which is perfectly revealed by π_t. Outcomes are identical to the full-information case $\pi_t = \frac{\phi_x+\sigma}{\phi_x+\sigma+\kappa\phi_\pi}u_t$ and $x_t = -\frac{\phi_\pi}{\phi_x+\sigma+\kappa\phi_\pi}u_t$. The determinacy condition differs from the one under full information, but in contrast to the first example still admits a unique equilibrium under limited information when the inflation coefficient is large enough, the output coefficient small enough and σ large enough. Notably, the condition $|\bar{\phi}_\pi| > 1$ does not depend on the Phillips curve slope.

Intuitively, in a full-information economy $|\bar{\phi}_\pi| > 1$ is tantamount to asking what would ensure determinacy if $x_t = -\frac{\phi_\pi}{\phi_x+\sigma}\pi_t$, which will always hold in the unique equilibrium of the original full-information economy. In other words, $|\bar{\phi}| > 1$ implies that the signal provided by inflation about the output gap is identical to the reduced-form outcome under full information, rather than leaving the analysis open for whatever other equilibrium paths might be consistent with the forward-looking Phillips curve.

If $|\bar{\phi}_\pi| < 1$ the rational expectations solution is indeterminate and may even be non-existent if a potential solution does not satisfy the projection condition. In the case of indeterminacy, belief shocks come into play, so that observed inflation need not perfectly reveal u_t.[10]

In general, the dynamic system representing this simple model can be characterized by the

[10]Based on the analysis of Lubik and Schorfheide (2003) there is room for only one belief shock since one of the roots of the linear RE system is inside, the other outside the unit circle.

following vector system:

$$\begin{bmatrix}\pi_{t+1}\\ x_{t+1}\end{bmatrix} = \frac{1}{\sigma\beta}\begin{bmatrix}\sigma & -\sigma\kappa\\ \beta\bar{\phi}-1 & \kappa+\beta\sigma\end{bmatrix}\begin{bmatrix}\pi_t\\ x_t\end{bmatrix} + \boldsymbol{\eta}_{t+1} + \begin{bmatrix}-\frac{1}{\beta}\\ \frac{1}{\sigma\beta}\end{bmatrix} u_t,$$

where $\boldsymbol{\eta}_{t+1}$ is a vector of endogenous forecast errors (that are yet to be determined). With $|\bar{\phi}| < 1$, the first matrix on the right-hand side has one root inside and one outside the unit circle so that any potential indeterminate solution is serially correlated. The projection condition is therefore a quadratic matrix equation. In general, analytical solutions are hard to obtain, except for some of the special cases considered in this section. We relegate a discussion of the numerical solution to the Appendix.

2.3 A Fisher Economy with Measurement Error

Our third introductory example moves away from the simple NK framework and uses the Fisher equation as the policy transmission channel. This allows us to introduce measurement errors into the framework while still maintaining analytical tractability. We consider measurement errors as another important source of limited information, especially with respect to the conduct of monetary policy. Initial data releases that serve as input to policy decisions are often revised later, so that a policymaker is often forced to project the likely underlying value of incoming data. We show with our example that this opens the door to equilibrium multiplicity.

The Fisher equation relates the nominal interest rate i_t to the real rate r_t and expected inflation $E_t\pi_{t+1}$:

$$i_t = r_t + E_t\pi_{t+1}. \tag{11}$$

The central bank implements monetary policy via the interest-rate rule:

$$i_t = \phi_\pi \pi_t.$$

We assume that the central bank observes only a noisy signal of inflation: $Z_t = \pi_t + \nu_t$, with $\nu_t \sim N\left(0, \sigma_\nu^2\right)$. Lacking knowledge of π_t, the policymaker forms conditional expectations $\pi_{t|t}$ and uses these in the policy rule instead: $i_t = \phi_\pi \pi_{t|t}$. In addition, for ease of exposition, we let the real rate be an $i.i.d.$ process with $r_t = \varepsilon_t$ and $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$. As before we assume that $|\phi_\pi| > 1$, so that the unique solution under full information is $\pi_t = \frac{1}{\phi_\pi} r_t$.

We obtain the rational expectations solution for the equation system conditional on the central bank's information set by noting that its projections of future inflation are identical to its projection of the public's current expectations about future inflation: $\pi_{t+1|t} = E\left(E_t\pi_{t+1} \mid Z^t\right)$ by the law of iterated expectations. Combining the Fisher equation with the policy rule then implies:

$$\pi_{t+1|t} = \phi_\pi \pi_{t|t} - r_{t|t}, \text{ and } r_{t+1|t} = 0,$$

which is isomorphic to the full-information case. Given our assumption on the policy parameter, the unique solution for the system in central bank projections is:

$$\pi_{t|t} = \frac{1}{\phi_\pi} r_{t|t}.$$

This projection condition restricts the joint behavior of central bank projections for inflation and the real rate, and is required to hold in a stable equilibrium (since the errors of the central bank's optimal projections are stationary so must be the projections when outcomes are stable). Any proposed equilibrium in the full equation system involving expectations of the private sector and the policymaker has to be consistent with what the central bank projects.

In contrast with the earlier examples, where the projections could be inferred directly from the assumptions on what is observed by the central bank, we now need to specify a projection equation, where the steady-state Kalman filter provides a recursive representation of the central bank's inflation projection:

$$\pi_{t|t} = (1 - \kappa_\pi)\, \pi_{t|t-1} + \kappa_\pi (\pi_t + \nu_t)\,, \text{ with } \kappa_\pi = \frac{cov\left(\pi_t, \pi_t + \nu_t \mid Z^{t-1}\right)}{var\left(\pi_t + \nu_t \mid Z^{t-1}\right)}.$$

The Kalman gain κ_π depends on the second moments of inflation, which in turn depend on the equilibrium dynamics. This in and of itself turns the solution of an otherwise standard linear rational expectations model into a non-linear problem. Since the projection condition implies that $\pi_{t|t-1} = 0$, the Kalman filter reduces to a static signal extraction problem with $\pi_{t|t} = \kappa_\pi (\pi_t + \nu_t)$. The policy rule then becomes $i_t = \phi_\pi \ \kappa_\pi (\pi_t + \nu_t)$. The relevant coefficient for characterizing the type of equilibrium is now $\phi_\pi \ \kappa_\pi$. In the appendix, we show that for equilibria where the real rate cannot be perfectly inferred, we have $|\phi_\pi \kappa_\pi| < 1$ for any ϕ_π, so that there is indeterminacy even if the Taylor principle holds (in the sense of $|\phi_\pi| > 1$).

The imperfect information setting thus delivers a stark implication similar to the one derived in the first example. Indeterminacy results from the endogenous attenuation of the incoming inflation signal when constructing the optimal projection. The policy rule depends on projected inflation, which in turn is given by the projection equation. Consequently, the sensitivity of the policy response to actual inflation depends on $\phi_\pi \ \kappa_\pi$, which itself is determined by the dynamics of inflation in equilibrium. Intuitively, a unique equilibrium might be obtained by setting the policy coefficient ϕ_π sufficiently high for a given Kalman gain κ_π. However, higher ϕ_π feeds back into stronger inflation stabilization and lower inflation variability, which reduces the signal-to-noise ratio of the central bank's signal. Optimal signal extraction therefore leads to a lower κ_π, which undoes the determinacy-creating effect from the more aggressive policy stance.

A crucial insight of our framework is thus that certainty equivalence does not hold despite the linear Gaussian setting.[11] Importantly, the Kalman gain depends on equilibrium outcomes for the second moments of inflation and the signal, which in turn depend on the Taylor-rule coefficients. In the Appendix, we present additional results and insights from this example. We show that it is possible to construct an equilibrium, where r_t is perfectly revealed, and $|\phi_\pi \kappa_\pi| > 1$. Similarly, we detail an alternative policy rule where the central bank responds directly to the noisy signal instead of the optimal projection based on the signal. We show that this policy rule leads to equilibrium determinacy. Finally, we also provide a full characterization of the equation system under limited information and its closed-form, analytical solution.

3 A General Framework

We now describe a general framework for analyzing linear rational expectations models with two information sets. We first review rational expectations equilibria (REE) under full information before turning to the imperfect information case, where one of the agents is strictly less informed than the other. The exposition in this section generalizes the simple examples presented in the previous section as to the type of information set. We keep the exposition largely descriptive and relegate the more technical details to the Appendix. This section can be safely skipped by readers mainly interested in the monetary policy applications.

3.1 A Linear Rational Expectations System with Two Information Sets

We consider a generic model economy with multiple agents that is described by the following system of linear expectational difference equations:

$$E_t \boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}} \boldsymbol{S}_{t+1|t} = \boldsymbol{A} \boldsymbol{S}_t + \hat{\boldsymbol{A}} \boldsymbol{S}_{t|t} + \boldsymbol{A}_i \, \boldsymbol{i}_t \,, \tag{12}$$

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_J \boldsymbol{S}_{t+1|t} + \boldsymbol{\Phi}_A \boldsymbol{S}_{t|t} \,, \tag{13}$$

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{X}_t' & \boldsymbol{Y}_t' \end{bmatrix}' . \tag{14}$$

$\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ are, respectively, vectors of backward- and forward-looking variables, while $\boldsymbol{i}_t$ denotes a vector of policy instruments that are under the control of one of the agents in the model.There are N_x backward- and N_y forward-looking variables and N_i policy instruments. The backward-looking

[11] This is of course not a new insight. It is well understood in the literature. e.g., at least since Sargent (1991) that certainty equivalence does not hold when imperfectly informed agents observe endogenous signals.

variables are characterized by exogenous forecast errors $\boldsymbol{\varepsilon}_t$:

$$\boldsymbol{X}_t - E_{t-1}\boldsymbol{X}_t = \boldsymbol{B}_{x\varepsilon}\,\boldsymbol{\varepsilon}_t\,, \qquad\qquad \boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I})\,,$$

where the number of independent shocks N_ε can be smaller than the number of backward-looking variables N_x, while $\boldsymbol{B}_{x\varepsilon}$ is assumed to have full rank. Furthermore, we assume that the initial value of the backward-looking variables $\boldsymbol{X}_0$ is exogenously given. Forecast errors for the forward-looking variables are endogenous and remain to be determined as part of the model's rational expectations solution as in Sims (2002):

$$\boldsymbol{\eta}_t \equiv \boldsymbol{Y}_t - E_{t-1}\boldsymbol{Y}_t.$$

The system of linear difference equations (12) - (13) captures the interdependent decision making of two types of agents, potentially derived from (linearized) optimality conditions of households, firms, or policymakers. All agents form rational expectations but they might condition on different information sets. We assume throughout that one type of agent has full information about the state of the economy. This could be, for instance a representative agent for the private sector, whose decisions are represented by (12) and also depend on the setting of a policy instrument $\boldsymbol{i}_t$ chosen by the other agent. In this example, the second agent is an imperfectly informed policymaker, who sets $\boldsymbol{i}_t$ according to the rule given in (13). By assumption, the policymaker knows the current value and history of the instrument choices, but lacks full knowledge of the state of the economy. Moreover, all variables entering the policy rule (13) are expressed as expectations conditional on the policymaker's information set, denoted $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$. The policymaker forms rational expectations based on an information set that is characterized by the observed history of a vector of N_z signals, denoted $\boldsymbol{Z}_t$, and knowledge of all model parameters. We assume common knowledge about the structure of the economy and all model parameters. We denote expectation formation as follows. For any variable $\boldsymbol{V}_t$, and any lead or lag h, $E_t\boldsymbol{V}_{t+h}$ is the rational expectation based on full information. Conditional expectations under the policymaker's information set are $\boldsymbol{V}_{t+h|t} \equiv E(\boldsymbol{V}_{t+h}|\boldsymbol{Z}^t)$, where $\boldsymbol{Z}^t = \{\boldsymbol{Z}_t, \boldsymbol{Z}_{t-1}, \boldsymbol{Z}_{t-2}, \ldots\}$.

We assume that the less informed agent observes a subset of model variables in a measurement vector with N_z elements, each of which is a linear combination of backward- and forward-looking variables:

$$\boldsymbol{Z}_t = \boldsymbol{H}\boldsymbol{S}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t\,. \tag{15}$$

In addition, the measurement vector can be affected by "measurement errors", which are disturbances to the measurement equation that would otherwise be absent from the full-information version of the model. We include any measurement errors in the vector of backward-looking variables, $\boldsymbol{X}_t$. By construction, policymaker actions $\boldsymbol{i}_t$ are spanned by the history of observed signals, such that trivially $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$. $\boldsymbol{i}_t$ merely reflects information contained in $\boldsymbol{Z}^t$ and thus need not be added to the description of the measurement vector.

We now briefly describe the equilibrium and solution to the full-information counterpart of our general framework. It is standard in the literature and rests on certainty equivalence and homogeneity of expectations. The equilibrium concept and solution approach for the imperfect-information framework is described in the next sub-sections. The interactions between different expectation formation processes require alternative solution procedures and lead to multiple equilibrium outcomes that abandon certainty equivalence. The general framework nests the case of full information, where $\boldsymbol{S}_{t+h|t} = E_t \boldsymbol{S}_{t+h}\ \forall h \geq 0$.

The full-information system can easily be solved using standards methods such as those found in King and Watson (1998), Klein (2000) or Sims (2002). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$:

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix}, \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_t \end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix} \boldsymbol{Y}_t \\ \boldsymbol{i}_t \end{bmatrix}. \tag{16}$$

Using $\boldsymbol{\mathcal{S}}_t' = \begin{bmatrix} \boldsymbol{i}_{t-1}' & \boldsymbol{S}_t' & \boldsymbol{i}_t' \end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} + \hat{\boldsymbol{J}} & 0 \\ 0 & \boldsymbol{\Phi}_J & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t \boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix} 0 & 0 & \boldsymbol{I} \\ 0 & \boldsymbol{A} + \hat{\boldsymbol{A}} & \boldsymbol{A}_i \\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{S}}_t \tag{17}$$

Existence and uniqueness of a solution to this system depend on the properties of the matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. Throughout this paper we focus on environments where a unique full-information solution exists. We therefore impose a standard root-counting criterion as in Klein (2000) and King and Watson (1998), namely that the number of explosive eigenvalues matches the number of forward-looking variables.[12] Specifically, we assume that the set of generalized eigenvalues of the system matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ as defined in (17) has $N_i + N_x$ roots inside the unit circle and $N_y + N_i$ roots outside the unit circle. The solution can be computed using the numerical methods of, for instance, Klein

[12] In the case of simple monetary policy models, the root-counting condition is satisfied by requiring that the central bank's interest-rate rule satisfies the Taylor principle, that is, by responding more than one-to-one to fluctuations in inflation.

(2000), and has the following form:

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}}\ \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}}\ \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}, \tag{18}$$

where $\boldsymbol{\mathcal{P}}$ is a stable matrix with all eigenvalues inside the unit circle. In the full-information version of our framework certainty equivalence holds; that is, the decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ nor on the measurement loadings $\boldsymbol{H}$. Equilibrium dynamics in the full-information case are then summarized by:

$$\boldsymbol{\mathcal{S}}_{t+1} = \bar{\boldsymbol{\mathcal{T}}}\boldsymbol{\mathcal{S}}_t + \bar{\boldsymbol{\mathcal{H}}}\boldsymbol{\varepsilon}_{t+1}\,, \qquad \bar{\boldsymbol{\mathcal{T}}} = \begin{bmatrix} \boldsymbol{\mathcal{P}} & 0 \\ \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}} & 0 \end{bmatrix}, \qquad \bar{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} \boldsymbol{B}_{x\varepsilon}\,, \tag{19}$$

where $\bar{\boldsymbol{\mathcal{T}}}$ is stable because $\boldsymbol{\mathcal{P}}$ is. The endogenous forecast errors are given by $\boldsymbol{\eta}_t = \boldsymbol{\mathcal{G}}_{yx}\,\boldsymbol{B}_{x\varepsilon}\,\boldsymbol{\varepsilon}_t$. One defining feature of this equilibrium is that non-fundamental disturbances, such as sunspot or belief shocks, do not affect equilibrium outcomes. Hence, the equilibrium is determinate, as discussed, for example, by Lubik and Schorfheide (2003). This is no longer the case under imperfect information.

3.2 A Class of Imperfect Information Equilibria

We now turn to the imperfect information case, where agents' expectation formation processes conditional on different information sets interact. We proceed by making a sequence of assumptions that allow us to map the general framework under asymmetric information into a standard linear rational expectations setting that can be solved and studied with familiar methods. Specifically, we are interested in linear equilibria, driven by normally distributed shocks. We show that the less-informed agent's expectations can be represented by a Kalman filter, which implies a linear relationship between projected innovations of unobserved variables and observed innovations in the signal received from the fully informed agent. It is the endogeneity of the filtering, which depends on the evolution of the economy, that gives rise to equilibrium multiplicity and a break-down of certainty equivalence under imperfect information. These results and insights with further discussion are contained in Appendix B.2.

The main result of this section, and key contribution of our paper, is that the equilibrium is generally indeterminate in this environment. Indeterminacy means that there is a multiplicity of initial conditions (or adjustment paths) to a long-run equilibrium (or steady state). An implication of indeterminacy is that along such adjustment paths the economy is subject to non-fundamental sunspot or belief shocks.

A key implication of our framework is that stationarity requires the mapping between projected backward- and forward-looking variables to be identical to the full-information case:

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}\,, \qquad \text{and} \qquad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t}\,. \tag{20}$$

where $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ are the unique solution coefficients in the corresponding full-information case.

We refer to the restrictions in (20) as "projection condition." As shown in the Appendix, the projection condition leads to a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$. The projection condition is a linear restriction on the covariances of $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$, or, equivalently the Kalman gains of forward- and backward-looking variables. It is this feature of our framework that moves the solution of the underlying linear rational expectations model out of certainty equivalence and poses an intricate fixed-point problem with a highly non-linear solution: The Kalman gains depend on the second moments of the solution, which in turn depends on the Kalman gains via the projection condition. As a second-moment restriction, the projection condition only restricts co-movements of the innovations *on average*, but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$.

We are now in a position to derive a representation of the system of expectational difference equations (12) - (14), where the expectations of the fully informed agent are represented by the concept of endogenous forecast errors as in Sims (2002) or Farmer, Khramov, and Nicolò (2015) and the expectation formation of the less-informed agent is represented by the Kalman filter. Equilibrium dynamics follow a state vector that tracks both projections and actual values of backward- and forward-looking variables as well as the policy instrument. More specifically, equilibrium dynamics are characterized by the evolution of the following vector system:

$$\overline{\boldsymbol{\mathcal{S}}}_{t+1} \equiv \begin{bmatrix} \boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}) & \boldsymbol{0} \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{C} & \boldsymbol{\mathcal{P}} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_t + \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H}) \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1} \end{bmatrix}\,, \tag{21}$$

where $\boldsymbol{\mathcal{K}}'_x = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{K}'_x \end{bmatrix}$, $\boldsymbol{C}$ as defined in the Appendix (see equation (A-62)), and $\boldsymbol{\mathcal{P}}$ known from the unique full-information solution.

Equation (21) forms the basis for solving a rational expectations model with heterogeneous information sets. Based on this representation we can derive key insights into the nature of the equilibria in this setting. The state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ follows a first-order linear difference system given in (21). The stability of the system depends on the eigenvalues of its transition matrix $\overline{\boldsymbol{\mathcal{A}}}$. The transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ depends on the Kalman gain $\boldsymbol{K}$, which depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$. Nevertheless, existence of a steady-state Kalman filter allows us to conclude that $\overline{\boldsymbol{\mathcal{A}}}$, is stable.

One key implication of this insight is that the usual root-counting arguments, as in Blanchard and Kahn (1980) or generalized in Sims (2002), do not pin down the shock loadings $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_{t+1}$ in (21). This follows from the fact that $\overline{\boldsymbol{\mathcal{A}}}$ is a stable matrix for any choice of $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ consistent with the existence of a steady-state Kalman filter. Moreover, the projection condition does generally not place sufficiently many restrictions on $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ to uniquely identify the shock loadings. This observation is the key result of our paper, namely that equilibria are usually indeterminate. Specifically, With $\overline{\boldsymbol{\mathcal{A}}}$ stable, the endogenous forecast errors are only restricted by the projection condition. The shocks loadings of the endogenous forecast errors on the structural shocks ε_t and belief shocks $\boldsymbol{b}_t$, $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$, have $N_y \times (N_\varepsilon + N_y)$ unknown conditions, while the projection condition imposes only $N_y \times N_z$ restrictions. As a result, the projection condition cannot uniquely identify the shock loadings $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$.

The standard macroeconomic literature on indeterminacy rests on the idea that forward-looking behavior is not restrained by a mechanism that pins down expectations. This mechanism is still present in our framework as it is embedded in the stability assumptions governing the solutions of the conditioned-down system and the existence conditions for the Kalman-filter. Intuitively, in the context of the introductory examples, the Taylor principle enforces uniqueness in a simple monetary model via an aggressive response to movements in inflation expectations, whether driven by fundamental or belief shocks. However, optimal filtering introduces, loosely speaking, too much stability in that it validates sunspot shocks as drivers of endogenous variables through the projection equation, which in turn affects expectation formation of the fully informed agent. We provide more discussion and simulation results from the simple models later in the paper.

We summarize the construction of an equilibrium for a given solution of the endogenous forecast errors in the following Theorem.

THEOREM 1 (Equilibria under Imperfect Information) *Consider the difference system characterized by 21 and let $\boldsymbol{\eta}_t = \mathbf{\Gamma}_\varepsilon \boldsymbol{\varepsilon}_t + \mathbf{\Gamma}_b \boldsymbol{b}_t$ with shock loadings $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ such that the projection condition (20) is satisfied. Equilibrium outcomes are then given by:*

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\mathcal{G}}_s \end{bmatrix} \overline{\boldsymbol{\mathcal{S}}}_t \qquad \boldsymbol{i}_t = \begin{bmatrix} 0 & \boldsymbol{\mathcal{G}}_i \end{bmatrix} \overline{\boldsymbol{\mathcal{S}}}_t \qquad \overline{\boldsymbol{\mathcal{S}}}_{t+1} = \overline{\boldsymbol{\mathcal{A}}}\ \overline{\boldsymbol{\mathcal{S}}}_t + \overline{\boldsymbol{\mathcal{B}}}\ \boldsymbol{w}_{t+1}\,,$$

$$\textit{with} \quad \overline{\boldsymbol{\mathcal{S}}}_t \equiv \begin{bmatrix} \boldsymbol{S}_t^* \\ \boldsymbol{\mathcal{X}}_t \end{bmatrix}, \qquad \boldsymbol{w}_{t+1} \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{b}_{t+1} \end{bmatrix}, \qquad \textit{and} \quad \overline{\boldsymbol{\mathcal{B}}} \equiv \begin{bmatrix} (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}) \\ \boldsymbol{\mathcal{K}}_{\mathcal{X}}\boldsymbol{D} \end{bmatrix}.$$

where $\boldsymbol{B}$ and $\boldsymbol{D}$ encode the shock loading $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ as stated in equation ((A-62)) of the Appendix; $\boldsymbol{\mathcal{K}}_x$, $\boldsymbol{\mathcal{G}}_s$ and $\boldsymbol{\mathcal{G}}_i$ are defined as above. Block matrices are partitioned along the lines of $\overline{\boldsymbol{\mathcal{S}}}_t$.

Proof. See Appendix. ■

Theorem 1 fully describes a solution to the linear rational expectations model with imperfect information and nested information sets. As a final step, we only need to provide a description of how the endogenous forecast errors, specifically the loadings on the shocks are determined. The Kalman gain $\boldsymbol{K}_x$ depends on the equilibrium distribution of endogenous forecast errors $\boldsymbol{\eta}_t$. By assumption, $\boldsymbol{\eta}_t$ is a linear combination of exogenous shocks $\boldsymbol{\varepsilon}_t$ and belief shocks $\boldsymbol{b}_t$ with endogenous shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ that have to satisfy the projection condition. In Appendix B.3 we describe our numerical approach to finding such shock loadings.

4 Quantitative Results

We now solve several limited-information models numerically and study the implications for equilibrium indeterminacy. The first set of quantitative results parallels the discussion in section 2, namely the simple NK model without measurement error, where the central bank observes inflation and output or alternatively only inflation. In neither case does the central bank observe the output *gap*. The second exercise solves the Fisher model numerically, while the third model environment is a richer NK model with measurement error in inflation and output.

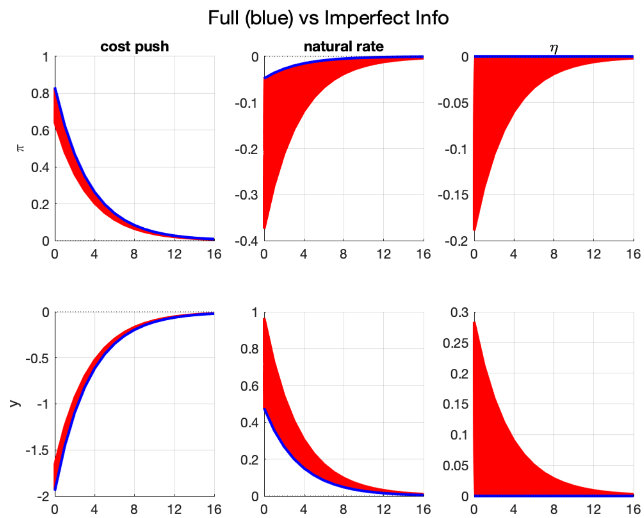
4.1 New Keynesian Models without Measurement Error

We first illustrate numerical solutions of the simple NK model described earlier. The central bank observes inflation and output, but not the output gap. There are two shocks: A shock to potential output, which we assume to be stationary, and a cost push shock. In contrast with the simple example specification in section 2 we use an AR(1) process for both shocks with AR-coefficient 0.75 and a standard deviation of 0.3 for the Gaussian innovation. All common parameters are the same as in our larger NK model in section 4.3. Further details on the calibration are found in that section.

Figure 1 shows the impulse responses under full information (in blue) and the set of impulse responses implied by the limited information equilibria (in red). Each column represents the response of inflation and output to a different shock. We normalize the sign impact of belief shocks; the mirror images of these responses are also valid impulse responses. Persistence arises exogenously here because of the driving process in contrast with the analytical example, where the solution was $i.i.d.$. The cost push shock raises inflation and lowers output, while the shock to potential output has the opposite effect. The responses to the former under limited information are relatively similar to those under full information, whereas the responses to the latter show initially larger amplification.

A noticeable feature of the responses is that the full-information solution is the envelope of the equilibria under limited information, which is a feature present in the other numerical results, too. In that sense, the full-information outcome is a limit of the limited information dynamics. This can

Figure 1: Impulse Responses in the NK Model with output and inflation observed by policymaker



Note: Full information outcomes in blue, limited information outcomes in red.

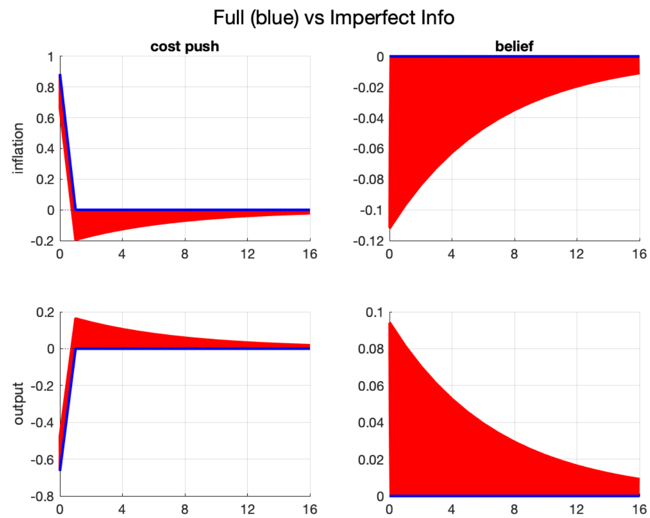
be shown analytically for some specifications in the presence of measurement error. Similarly, in the Appendix we discuss an equilibrium under limited information that is identical to the corresponding one under full information (subject to the presence of belief shocks). The second noticeable feature is that the set of indeterminate equilibria is restricted by the projection condition, which bounds, for example, the belief shock responses.

Figure 2 reports the responses from the second NK example, where only inflation is observed and the cost-push shock is $i.i.d.$. As discussed in section 2, in this case persistence arises endogenously because the Kalman filtering process adds stable dynamics to the equation system, whereas under full information all forward-looking dynamics are explosive. The cost-push shock raises inflation (and lowers output), which is reversed after one period because the policy rate increases, but not by enough to fully stabilize the endogenous variables. Moreover, the projection corrects for the initial strong observed inflation response. As in the previous example, the set of responses, namely the set of factor loadings for each possible equilibrium, is limited by the projection condition.

4.2 The Fisher Economy

We now turn to a Fisher economy and consider the case of the policy rule $i_t = \phi \pi_{t|t}$, where the endogenous information set $Z_t = \pi_t + \nu_t$. For purposes of illustration, we set the policy parameter $\phi = 1.5$ and assume that the real rate follows an AR(1) process with persistence $\rho = 0.9$ and a unit innovation variance $\sigma_\varepsilon^2 = 1$.The variance of the belief shock σ_b^2 is normalized to unity without loss

Figure 2: Impulse responses in simple NK model when policymaker observes only inflation



Note: Full information outcomes (blue) as well as various limited information equilibria (red).

of generality. We set the variance of the $i.i.d.$ measurement error ν_t to $\sigma_\nu^2 = 2.5^2$. Each equilibrium is associated with a triple $(\gamma_\varepsilon, \gamma_b, \gamma_\nu)$ of loadings on the shocks. The fact that there are multiple such loadings for the parameter space simply reflects that the rational expectations solution is indeterminate.

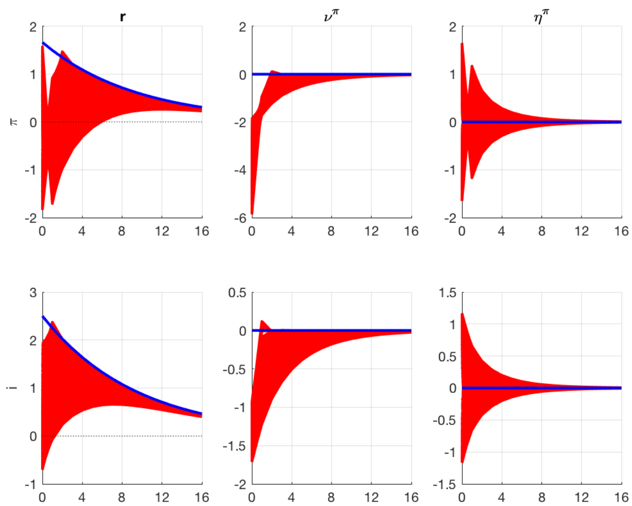
We plot the set of impulse responses to each shock obtained over the entire set of equilibria in Figure 3. We select an arbitrary example equilibrium, and show its impulse responses in Figure 4. Each figure also displays the corresponding full-information responses. In the latter case, a unit innovation to the real rate raises inflation by $1/(\phi - \rho) = 5/3$ which then decays at the constant rate ρ. The interest response follows the same pattern, which reflects the Fisher effect in that a higher real rate requires a higher nominal rate and in turn a higher inflation rate.[13]

Under limited information the set of equilibria is notably different. Belief shocks play a role and multiple values of shock loadings are possible. On impact, a unit innovation in the real rate can either lead to an increase or a decrease in inflation over a range of about $(-1.9, 1.7)$ depending on the specific equilibrium the economy is in. Similarly, the nominal rate response can be positive or negative. In effect, inflation and the nominal rate can comove positively or negatively under different equilibria.[14] Figure 4 displays impulse response for one of the possible limited information

[13]The unique full-information equilibrium has zero response to the measurement error since the model is not defined as having such error, and there is also a zero-response to the belief shock since the solution is unique with $\phi = 1.5$.

[14]The findings are reminiscent of the observation by Lubik and Schorfheide (2004) that changes in comovement patterns are a hallmark of equilibrium indeterminacy and thereby allow econometricians to identify different sets of equilibria. Moreover, their observation that indeterminate equilibria do impose some restrictions on the behavior of the

Figure 3: IRFs of Various Equilibria in the Fisher Model



Note: Impulse response functions (IRF) under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. The column labeled η^{π} reports responses to belief shocks.

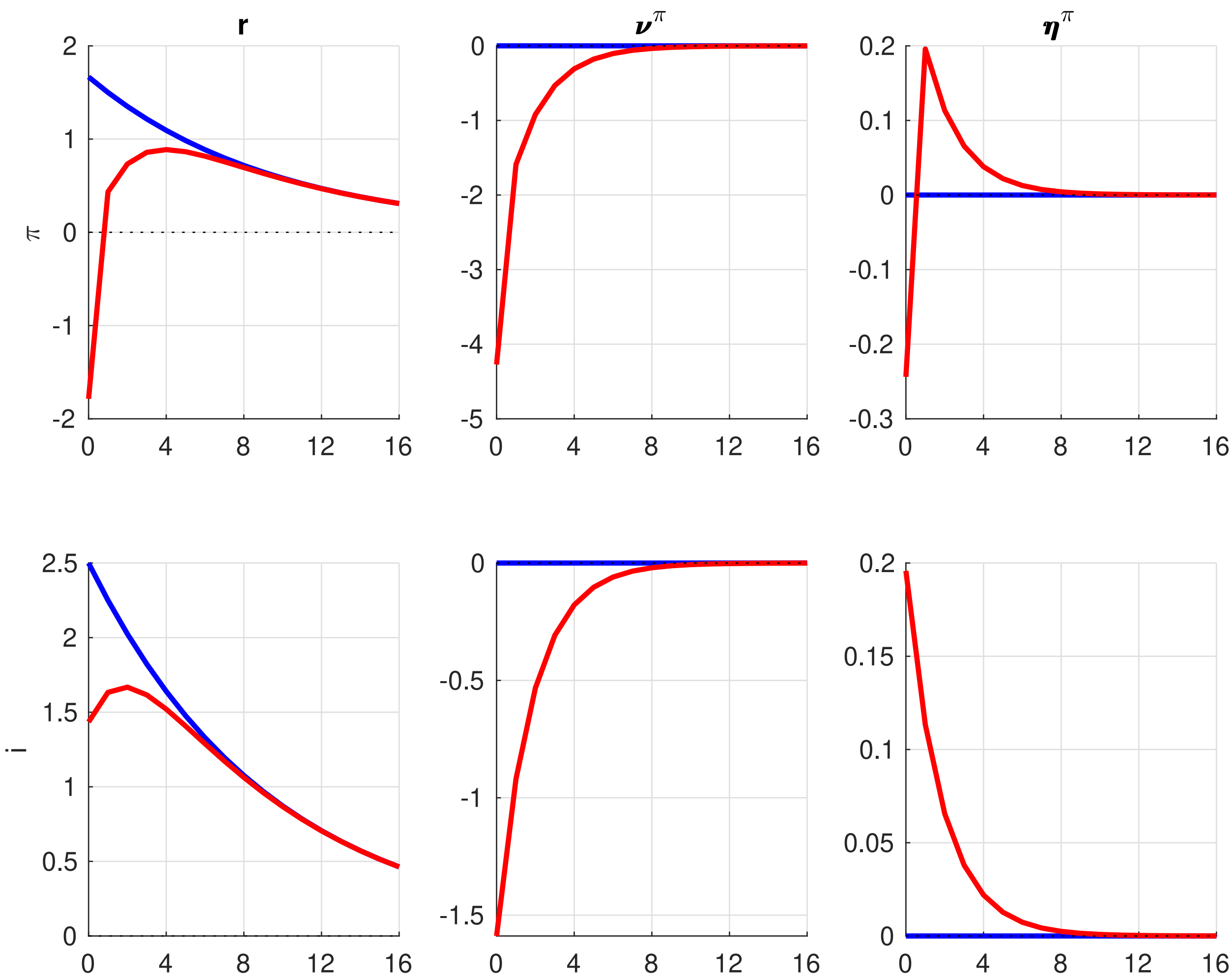
equilibria where, in contrast to the full-information solution, inflation and the nominal interest rate comove negatively. While the nominal rate follows the real rate increase, inflation can fall on account of a negative loading γ_{ε} on the forecast error. The figure also shows that a unit measurement error shock lowers inflation and the nominal rate, which indicates a negative loading on ν_t in the solution, $\gamma_{\nu} < 0$. In addition, belief shocks play a non-trivial role. Critically, the range of possible responses to the belief shock across equilibria shown in Figure 1 is bounded on account of the projection condition.

Figure 5 reports the autocorrelation function (ACF) and the standard deviation relative to the full information scenario.[15] As shown in the upper row of panels, the full-information solution displays the typical autocorrelation pattern of a first-order autoregressive process. Comparing the set of outcomes under limited information against the full-information equilibrium, the persistence of inflation is generally lower and its serial correlation decays much more rapidly, whereas for the nominal rate the ACF closely resembles that under full information. Since $i_t = \phi\pi_{t|t} = \frac{\phi}{\phi-\rho} r_{t|t}$ the nominal rate behaves like the real rate projection. The lower panel of the figure shows ranges of relative standard deviations of outcomes under limited information relative to the respective full-

economy in response to fundamental shocks thus carries over to our framework.

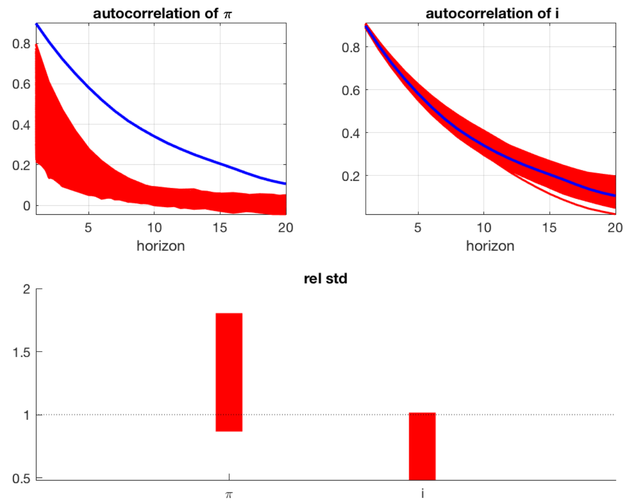
[15] Moments are computed via simulation for 20,000 periods with the first 1,000 periods discarded as burn-in to avoid dependence on initial conditions.

Figure 4: IRF from an Arbitrarily Selected Equilibrium in the Fisher Model



Note: Impulse response functions (IRF) under full information (blue) as well as an example from the limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. The column labeled η^{π} reports responses to belief shocks.

Figure 5: Second Moments of Limited-Information Equilibria in the Fisher Model



Note: Top panels show moments of endogenous variables for the simple example economy under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes.

information outcomes. The interest rate under limited information is generally less volatile despite the presence of two additional shocks. The lower interest-rate volatility echoes our discussion in section 2 that the *effective* response to inflation under limited information after taking into account the filtering problem constitutes a violation of the Taylor principle. Imperfect information prevents the central bank from moving the policy rate aggressively in response to actual inflation. Instead, optimal filtering leads to attenuation of the policy response, the flipside of which is heightened inflation volatility.

We conduct two robustness checks to study how the properties of equilibrium change when we vary key parameters, First, we show in Figure 6 how our equilibrium outcomes depend on the volatility of the measurement error. We consider two cases: an increase and decrease of its standard deviation by 10 percent. For each specification, we record the minimum (in red) and maximum (in green) responses of the variables to the shocks at each horizon *across equilibria*. The figure shows that an increase in the measurement error variance tends to increase the possible values the impulse response can take. This is unequivocally true in the case of belief shocks. Figure 7 shows how impulse responses change when we vary the response coefficient in the monetary policy rule. The red set of responses correspond to a decrease in the coefficient relative to the benchmark case, whereas the green line corresponds to an increase. Note that in this case the full information response changes as well.

4.3 A Small-Scale NK DSGE Model with Measurement Error

We now show quantitative results for a small-scale NK DSGE model where there is measurement error. Similar to the simple example NK model, we assume that there is a private sector that has the same information set as households in the full information version of the model. We also assume that the central bank only observes noisy measurements of inflation and the level of real GDP, which implements the limited information component by introducing measurement error as in the third example of section 2. Finally, the central bank follows a monetary policy rule with which responds to its best estimate of inflation and the output gap. In contrast to the simple NK model we studied earlier, we now assume (i) that there is a backward-looking component in inflation, and (ii) that potential GDP is non-stationary (we posit a stationary process for its first difference).

We specify a version of the NK model that includes a Phillips curve with a backward-looking component in inflation π_t:

$$(1-\gamma\beta)\,\pi_t = \beta E_t \pi_{t+1} + \gamma \pi_{t-1} + x_t, \tag{22}$$

where $0 \leq \gamma < 1$ denotes the degree of indexation and governs inflation persistence. x_t is the output gap and the sole driver of inflation. Its evolution is captured by a variant of the Euler-equation, which relates output to the real rate and policy actions:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \bar{r}_t\right), \tag{23}$$

where $\sigma > 0$ is the intertemporal substitution elasticity and governs the responsiveness of output growth to interest rate movements. The term in parentheses is the gap between the actual real rate of interest $(i_t - E_t \pi_{t+1})$ and its natural rate r_t. Similar to Laubach and Williams (2003), we assume that r_t is related to expected growth in potential real GDP $\bar{y}_t$:

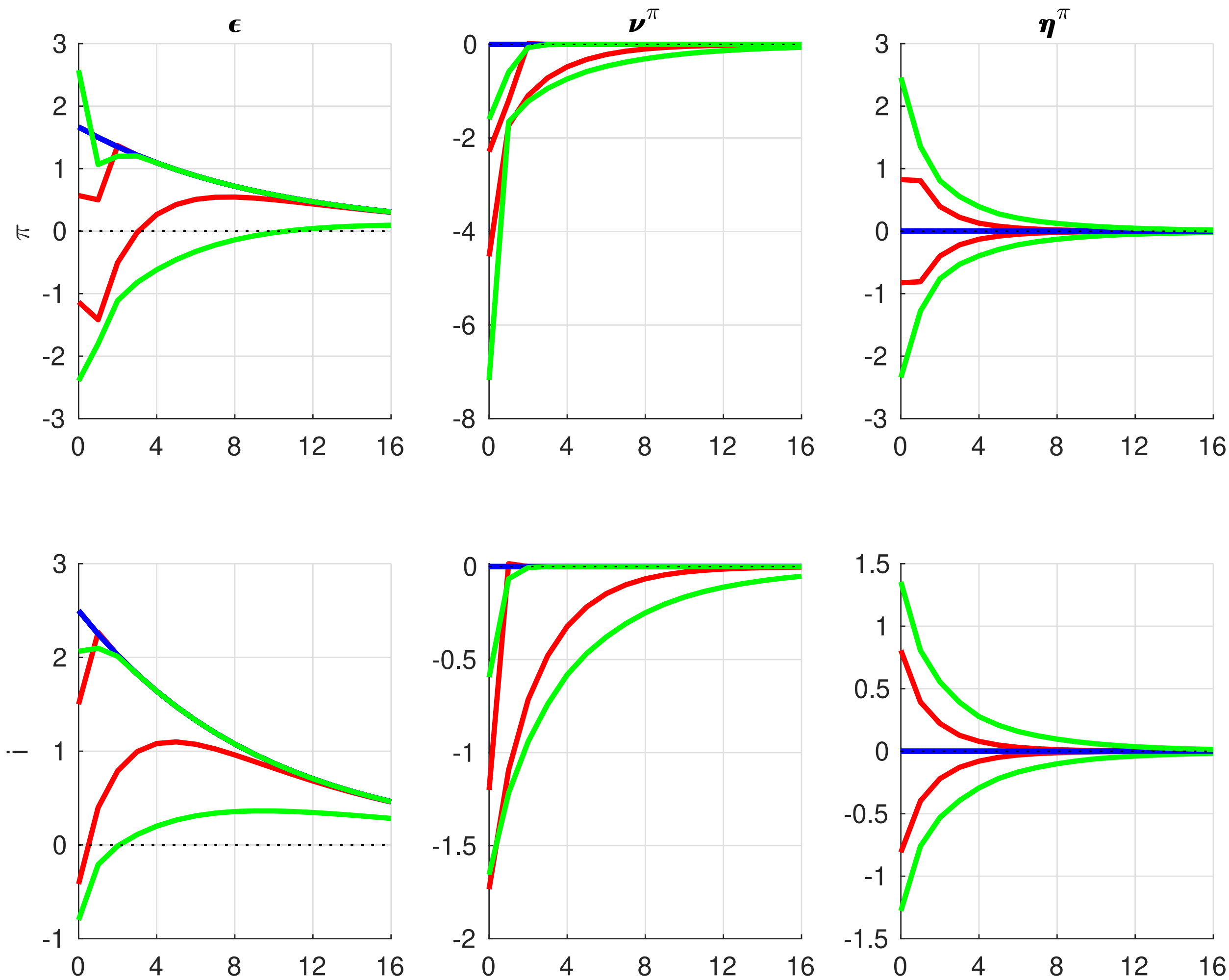
$$\bar{r}_t = \sigma E_t \Delta \bar{y}_{t+1}.$$

Furthermore, we assume that $\Delta \bar{y}_t$ follows an autoregressive process of order one:

$$\Delta \bar{y}_t = \rho_y \Delta \bar{y}_{t-1} + \varepsilon_t^y,$$

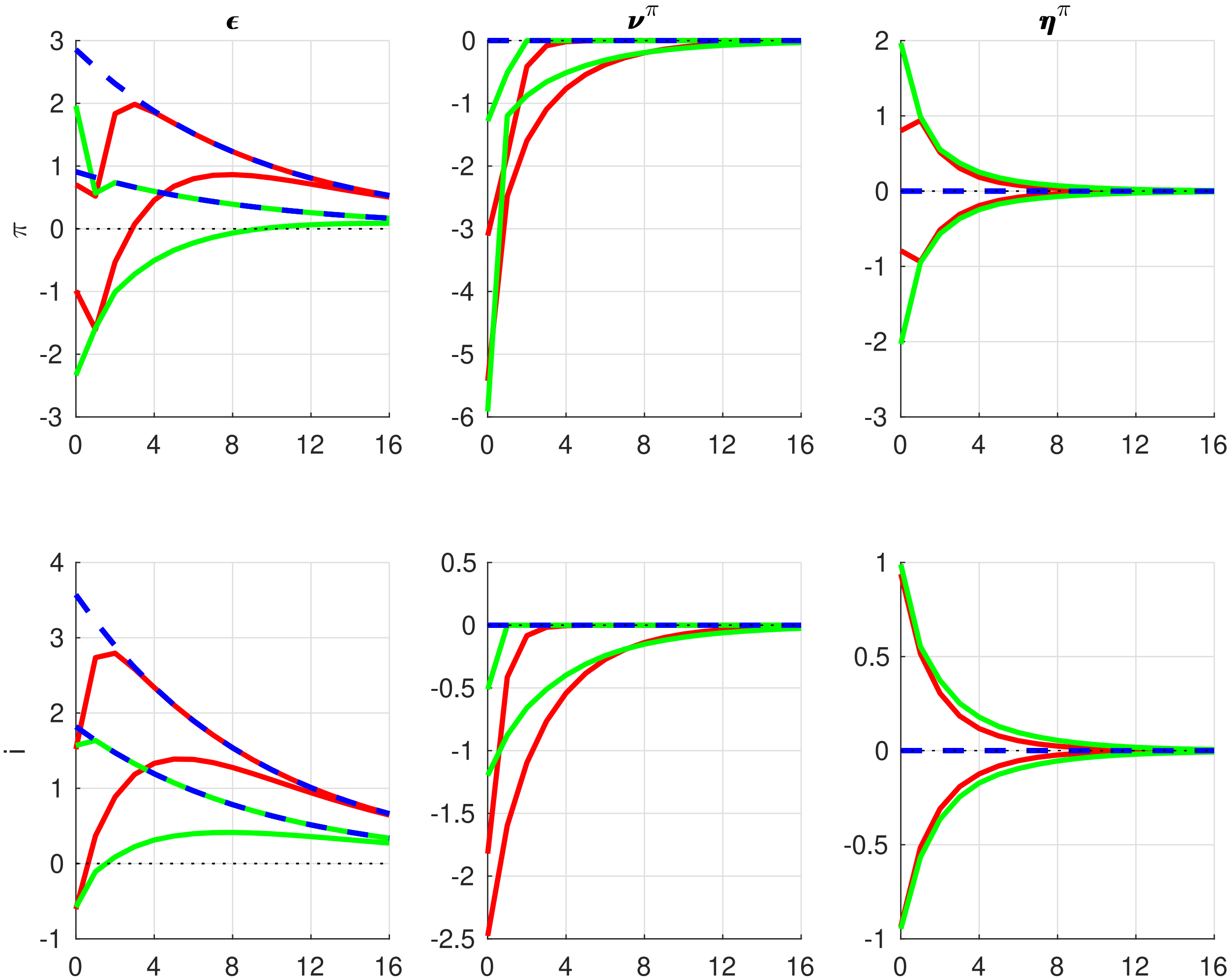
where the innovation $\varepsilon_t^{\bar{y}}$ is $i.i.d.$ Gaussian with zero mean and finite variance $\sigma_{\bar{y}}^2$, $\varepsilon_t^{\bar{y}} \sim N\left(0, \sigma_{\bar{y}}^2\right)$.

Figure 6: Effect of changes in the volatility of the measurement error on impulse responses in the Fisher model.



Note: Plots the minimum and maximum responses at each horizon across equilibria for different measurement error standard deviations. Results for a ten percent decrease are in red, results for a ten percent increase are in green. Full information outcomes are in blue.

Figure 7: Effect of changes in policy-rule coefficients on impulse responses in the Fisher model.



Note: Plots the minimum and maximum responses at each horizon across equilibria for different monetary policy rule coefficients. Results for 1.25 are in red, results for 2.0 are in green. Full information outcomes are the blue dashed lines.

Table 1: Parameters for NK model

Symbol	Description	Value
β	Discount Factor	0.99
σ	Substitution Elasticity	1.00
ϕ	Labor Elasticity	1.00
γ	Inflation Indexation	0.25
ϕ_π	Policy Coefficient	2.50
ϕ_x	Policy Coefficient	0.50
$\rho_{\bar{y}}$	AR(1) - Coefficient	0.75
$\sigma_{\bar{y}}$	StD. Output Growth	0.30
σ_π	StD. Measurement Error	0.80
σ_x	StD. Measurement Error	1.39
κ	Composite Parameter	0.17

Note: Parameter values for the numerical analysis of the NK model. Values are standard in the literature.

The central bank follows the feedback rule:

$$i_t = \phi_\pi \pi_{t|t} + \phi_x x_{t|t}, \tag{24}$$

where we assume that the policy coefficients ϕ_π and ϕ_x are such that in the full-information version of this model the equilibrium is unique. As before, $x_{t|t}$ denotes the output gap projection given information available to the central bank at time t. We choose this specification as it arguably mirrors more closely the practice of many central banks, including the Federal Reserve.

We introduce two measurement errors in inflation and the level of output, ν_t^π and ν_t^x, respectively.[16] The measurement errors are jointly normally distributed and serially and mutually uncorrelated with variances σ_π^2 and σ_x^2. This specification implies the following measurement vector:

$$\boldsymbol{Z}_t = \begin{bmatrix} \pi_t + \nu_t^\pi \\ y_t + \nu_t^x \end{bmatrix}, \text{ with } \begin{bmatrix} \nu_t^\pi \\ \nu_t^x \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix}\right).$$

We calibrate the model by choosing standard parameter values in the literature (see Table 1). We set the intertemporal substitution elasticity to $\sigma = 1$ to maintain comparability with the simple example, while $\beta = 0.99$. The indexation parameter is chosen as $\gamma = 0.25$, which roughly replicates observed inflation persistence. The policy coefficients ϕ_π and ϕ_x are set to 2.5 and 0.5 which guarantees the existence of equilibria in a wide neighborhood of the parameterization. We

[16]The level of GDP, y_t, is by construction equal to the growth rate in potential GDP plus the sum of lagged potential GDP and the current output gap, so that $y_t = \Delta\bar{y}_t + \bar{y}_{t-1} + x_t$.

calibrate the measurement error processes in line with the empirical findings in Lubik and Matthes (2016). For inflation, we choose the standard deviation of our measurement error to match their estimated unconditional standard deviation for the inflation error. For real GDP, we assume that log GDP is measured with $i.i.d.$ error. This automatically induces autocorrelation in the measurement error for the log-difference of GDP, consistent with their findings.

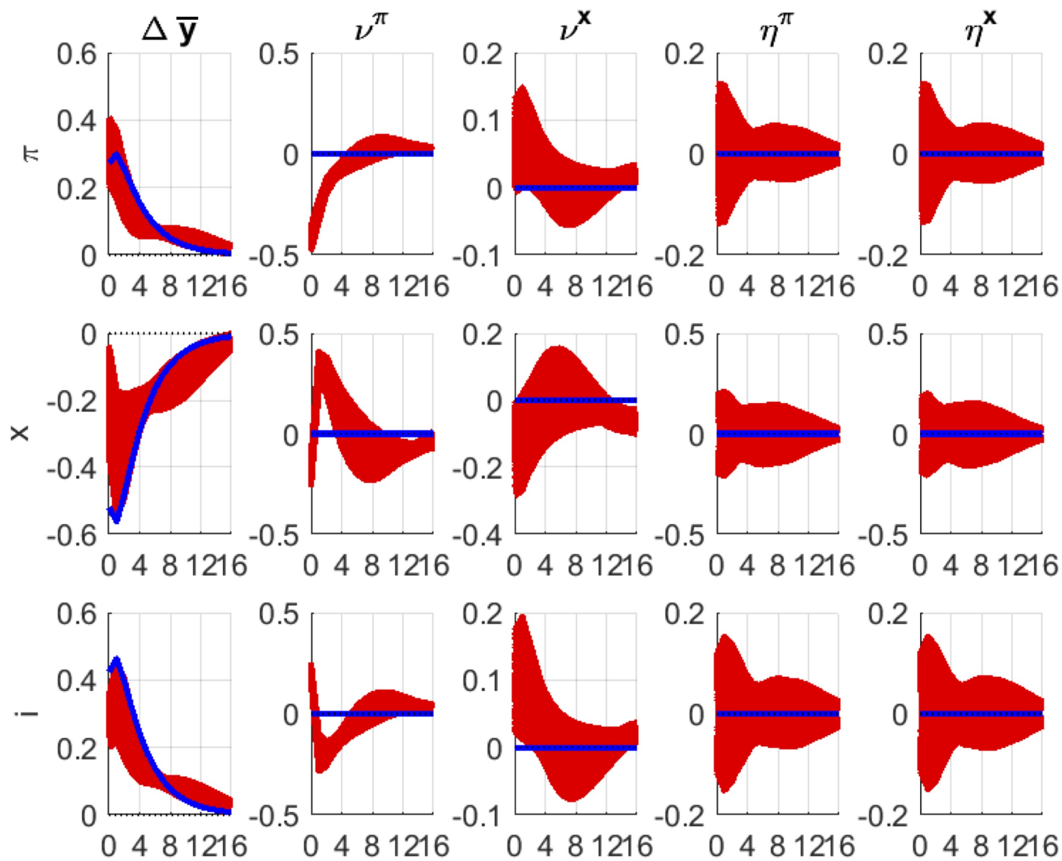
Figure 8 reports impulse response functions for the shock to potential GDP growth in the left column, next to the two measurement errors and the two belief shocks, denoted η^{π} and η^{x}. Each row shows the response of the model's endogenous variables. Solid blue lines indicate the responses under full information, while the areas in red capture the responses for different equilibria under limited information. In the latter case, the different responses are driven by different shock loadings γ, which are not uniquely determined in equilibrium. As the only shock that affects outcomes under full information the innovation to potential GDP growth increases the natural real rate via the expectations channel. This prompts a rise in the policy rate and reduces expected output gap growth on impact due to a fall in the current output gap. As actual production ramps up to close the gap, inflation declines from its initial peak, which is driven by the relative reduction in supply on impact.

The impulse responses to the fundamental innovation in the limited information case are qualitatively similar to the previous responses although they show a somewhat richer dynamic adjustment pattern. Figure 9 shows responses for a single limited information equilibrium that are fairly close to their full information counterparts. We also note that the full-information response is in parts an envelope to the area of responses associated with indeterminate equilibria, a pattern we also observe in the simple models.

The next two columns in Figure 8 show the responses to the measurement error shocks. A positive shock to ν_t^{π} prompts the central bank to adjust its inflation projection upwards. This stimulates a contemporaneous rise in the policy rate and generally lowers the output gap due to a fall in current GDP. The inflation rate falls because of the contractionary central bank policy response to the inflation mismeasurement. This pattern is also evident from Figure 9. However, there are equilibria where this pattern is overturned with a considerably smaller, even negative, interest rate response. What drives these differences are the different values of the endogenous Kalman gain associated with various indeterminate equilibria. That is, there are equilibria, in which the responsiveness of inflation projections to measurement error is small enough so that the standard adjustment dynamics in response to output gap movements and their projections dominate. Responses to the output measurement error follow a similar but less pronounced pattern. A positive innovation ν_t^{x} leads to an upward revision of output gap projections and an interest rate hike, followed by a decline in current output and a rise in prices. Adjustment patterns to both measurement errors exhibit slowly adjusting and oscillating dynamics.

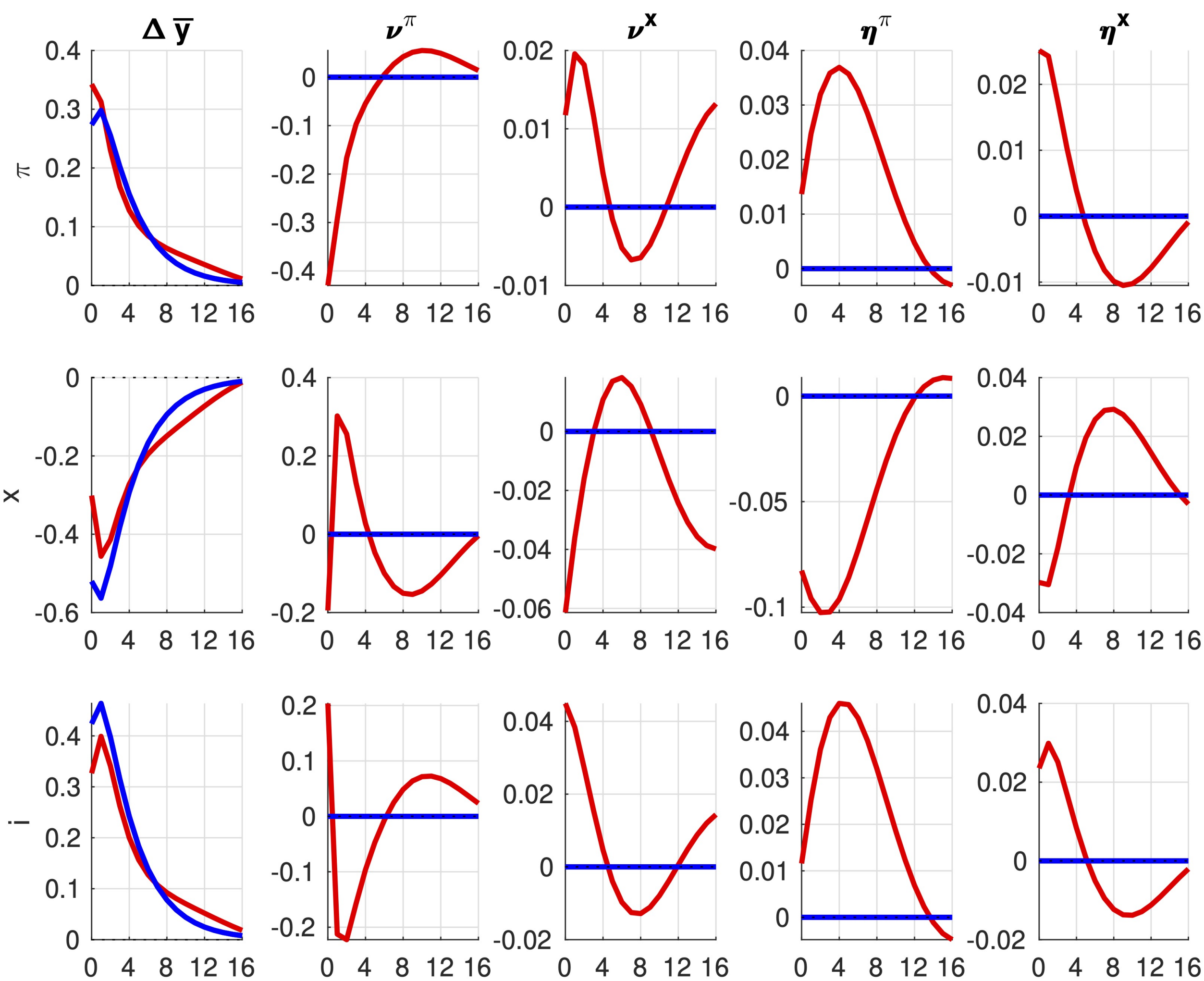
Finally, the last two columns in Figure 8 show the responses to the belief shocks, which

Figure 8: IRFs of Various Equilibria in NK model



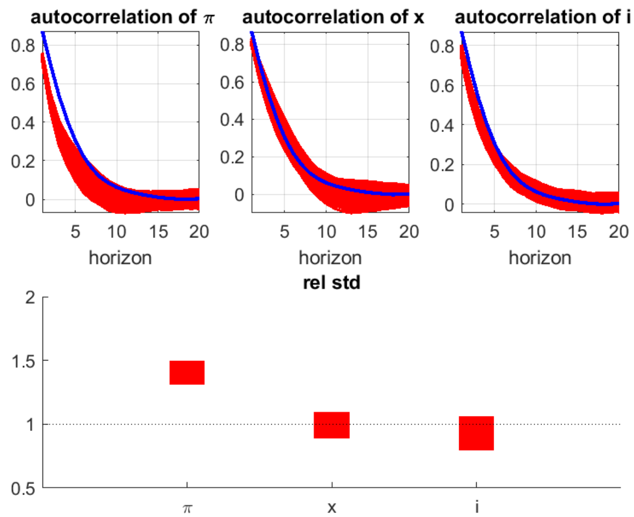
Note: Impulse response functions (IRF) for the NK model under full information (blue) as well as various limited-information equilibria (red). Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock. An example of the IRF of one of the limited-information equilibria is show in Figure (9).

Figure 9: IRF in NK model: Example of a limited information equilibrium



Note: Impulse responses for one example (red) of the limited-information equilibria of the NK model shown in Figure 8. (Full-information IRF in blue.)

Figure 10: Second moments of limited-information equilibria in NK model



Note: Top panels show moments of endogenous variables for the NK model under full information (blue) as well as various limited information equilibria (red). Bottom panel reports ranges of relative standard deviations of outcomes under limited information relative to the full-information outcomes.

are identical and symmetric.The set of impulse response functions is symmetric around the zero line since the response to a sunspot shock in each equilibrium is only determined up to its sign. Nevertheless, we can still trace out the effect of, for instance, a positive belief shock to inflation such as in the fourth column of Figure 9, where we plot the responses for one specific equilibrium. Suppose consumers believe inflation to be higher than initially anticipated, the belief being driven by the realization of a sunspot that is interpreted as fundamental. This leads the central bank to raise its inflation projection somewhat, but not fully given its filtering problem. The policy rate rises, but not to the full extent required to invalidate consumers' beliefs. This would occur if the policy response were such that it raised the real rate by enough to reign in increased spending and thus rising prices. Although the central bank obeys the Taylor principle, the wedge between private sector and central bank expectations generated by the filtering process is sufficient for indeterminacy to arise. The less than aggressive interest-rate response thereby leads to output gap movements that validate beliefs to the extent that inflation rises by enough consistent with the initial belief.

Similar to the quantitative results for the simple examples above, we compute autocorrelation functions and relative standard deviations for the NK DSGE model, which are reported in Figure 10. The upper panel shows the ACFs for the three key variables in the model. What is notable is that the ACFs cluster around the corresponding ACF under full information. A similar impression is conveyed by the range of relative standard deviations in the lower panel. We find the same pattern as in the simple models, namely higher inflation volatility and a slightly lower interest rate volatility, which reflects the less aggressive policy response under full information.

4.4 Summary and Interpretation

In standard NK models such as Clarida, Gali, and Gertler (2000) or Lubik and Schorfheide (2004) indeterminacy arises because the central bank conducts a policy that does not satisfy the Taylor principle. The Taylor principle prescribes a sufficiently strong response of the nominal policy rate to actual inflation. In our limited information setting, the central bank applies the Taylor principle with respect to projections derived from an optimal filter. Since the optimal filter attenuates the response to its incoming inflation signal, the policy response to actual inflation ends up being too weak to ensure determinacy. Deviations from the Taylor principle lead to sunspot-driven movements in private sector expectations that the central bank cannot invalidate through its actions. Even though there is a unique mapping between central bank projections of outcomes and economic conditions in our framework, actual outcomes remain indeterminate. The source of the indeterminacy is the interaction of expectations formed under the two information sets.

When outcomes are not uniquely determined by economic fundamentals, there is a role for belief shocks to drive economic fluctuations. The term "belief shocks" refers to a set of economic disturbances that matter since people believe that they do. In general, these disturbances are otherwise unrelated to economic fundamentals.[17] We can think of the implications of belief shocks in terms of the following thought experiment. Suppose that the realization of a sunspot leads the private sector to believe that inflation is higher than warranted by economic fundamentals. This implies a reassessment of the nominal interest-rate path and a higher i_t in compensation for higher expected inflation. At this point, the behavior of the central bank is crucial. In the full-information model, if the Taylor principle holds, the central bank would raise the policy rate by proportionally more than the private-sector's sunspot-driven increase in inflation. This is only consistent with the Fisher equation if expected inflation rose by more than current inflation. However, this would imply an explosive trajectory for inflation, which is ruled out as an equilibrium path. If the Taylor principle does not hold, the policy rate rises less than the sunspot-driven increase in current inflation. In this case, the Fisher equation implies a less than proportionate and thereby stable increase in expected inflation, which validates the sunspot (or belief shock) to be consistent with a stationary equilibrium. Consequently, the resulting equilibrium can be subject to belief shocks and is indeterminate.

The logic of belief shocks applies to the case of measurement error as an example for information limitations as well as restricted information sets as in the case of the first two simple examples in section 2. In our limited information setup, the policy rule cannot apply the Taylor principle

[17]The use of the term "beliefs" is conceptually distinct from the "projections" described as part of our imperfect information setup, where projections are the result of the policymaker's optimal signal extraction efforts.

to actual inflation, which is unobserved. Instead, the policymaker applies the disproportionate response recommended by the Taylor principle to an estimate of current inflation. The estimate solves a signal extraction problem that attenuates the noisy signal of inflation so that policy ends up responding less than proportionally to actual inflation. As discussed in section 2, making the policy response to projected inflation more aggressive does not restore determinacy as it also attenuates the effect of actual inflation on the policymaker's optimal inflation projection. However, consistency of expectations held by the private sector and the imperfectly informed policymaker places a bound on the contribution of belief shocks to inflation.

5 Conclusion

This paper studies the implications of limited information for equilibrium determination in linear dynamic models when differently informed agents interact. We introduce a single deviation from full information rational expectations: one group of agents is strictly less informed than another. While we differentiate between types of agents that have different information sets, we continue to assume that each agent forms rational expectations conditional on available information. The implications of this model structure are stark. We show that indeterminacy of equilibrium is widespread in this environment, even if the corresponding full information setting implies uniqueness. A given amount of noise, with optimal filtering of a less informed agent, can produce outcomes where there is a sunspot component to economic fluctuations. Our paper thereby contributes to a recent literature on informational frictions in rational expectations models.

Throughout our analysis, we maintain the assumption that the policymaker's limited information set is nested inside the public's information set. The indeterminacy issues identified by our paper should, however, extend to richer informational environments as long as the policymaker cannot perfectly observe forward-looking choice variables of the private sector. The key condition behind our indeterminacy results is that imperfect information impedes the policymaker from directly responding to belief-shock induced variations in private-sector decisions, which would also hinder policy in a setting of non-nested information.

The findings in this paper suggest various avenues for further investigation. For example, our framework has strong implications for empirical research: The general model under limited information has a state-space representation like any other linear dynamic framework so that a likelihood function can be constructed. The key difference and main complication with respect to standard frameworks is that the solution of the model is not certainty equivalent. Conditional on the Kalman gains the model implies a standard representation, but the gains are equilibrium objects and depend on second moment properties of the solution. This can be taken into account in solution and estimation, albeit at the cost of posing non-trivial computational challenges.

While we apply our framework to various models with an imperfectly informed central bank, it is not limited to applications in monetary policy. In fact, section 3 shows that indeterminacy is a feature of a general class of economies where private-sector behavior is characterized by a set of expectational linear difference equations, exogenous driving processes are Gaussian, policy is described by a linear rule that responds to the policymaker's projections of economic conditions, and the projections are rational. A related issue is the choice of the information set. In our example, we endow the central bank with specific information sets. Alternatively, one could imagine a scenario where the policymaker chooses an optimal information set that minimizes the impact of sunspot shocks and possibly reduces the incidence of multiplicity. This direction has relevance for policy as central banks operate in a real-time environment fraught with measurement error and regularly face judgment calls on the importance of incoming data. An important extension should be to look beyond a given class of linear policy rules, as considered here, and model the optimal policy choice for a given set of preferences. Such an exercise could also consider how a desirable policy could be implemented with a suitable policy rule, which requires an analysis of equilibrium selection in the presence of indeterminacy.

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Online Appendix to "Indeterminacy and Imperfect Information"*

Thomas A. Lubik
Federal Reserve Bank of Richmond[†]

Christian Matthes
Indiana University[‡]

Elmar Mertens
Deutsche Bundesbank[§]

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Contents

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[†]Research Department, P.O. Box 27622, Richmond, VA 23261. Email: thomas.lubik@rich.frb.org.

[‡]Department of Economics, Wylie Hall 202, Bloomington, IN 47405. Email: matthesc@iu.edu.

[§]Research Centre, Wilhelm-Epstein-Str. 14, 60431 Frankfurt, Germany. Email: elmar.mertens@bundesbank.de.

A Three Motivating Examples: Further Details

A.1 NK Model with Observed Output and Inflation

We assume that the central bank's information set is $Z_t = \{\pi_t, y_t\}$, which implies that $\pi_{t|t} = \pi_t$ and $y_{t|t} = y_t$. However, the central bank still has to project $x_{t|t}$, $\overline{y}_{t|t}$, and $u_{t|t}$. We derive these projections from the conditioned-down system, namely the solution of the model under the central bank's information set. This is isomorphic to the full-information solution. Specifically, as noted in equations (5) – (7) of the paper:

$$\pi_{t|t} = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\kappa\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{5}$$

$$x_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} - \frac{\sigma}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}, \tag{6}$$

$$y_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t} + \frac{\phi_x + \kappa\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} \overline{y}_{t|t}. \tag{7}$$

Together with $\pi_{t|t} = \pi_t$ and $y_{t|t} = y_t$, this equation system can be solved for the projections as functions of the variables in the central bank's information set:

$$\begin{aligned}
\overline{y}_{t|t} &= \frac{\phi_x + \sigma}{\phi_x} y_t + \frac{\phi_\pi}{\phi_x} \pi_t, \\
u_{t|t} &= \frac{\kappa\sigma}{\phi_x} y_t + \frac{\phi_x + \kappa\phi_\pi}{\phi_x} \pi_t, \\
x_{t|t} &= -\frac{\sigma}{\phi_x} y_t - \frac{\phi_\pi}{\phi_x} \pi_t.
\end{aligned}$$

This expression can be used to find a 'reduced-form' policy rule by substituting out $x_{t|t}$:

$$i_t = -\sigma y_t.$$

Using this rule in the original equation system results in the following system:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + u_t, \\
0 &= E_t x_{t+1} + E_t \pi_{t+1}.
\end{aligned}$$

It can be quickly verified that this system is singular, that is, underdetermined. Consequently, and in addition, the solution is indeterminate since the second equation, which is derived from the Euler-equation after imposing the reduced-form policy rule, only pins down (linear combinations of) expectations of the output gap and inflation and not their actual values. In contrast, the first equation is an unstable difference equation ($0 < \beta < 1$) and determines inflation dynamics. Using the forecast-error decomposition from Sims (2002), we can write:

$$\begin{aligned}\pi_t &= \beta\pi_{t+1} + \kappa x_t + u_t - \beta\eta^{\pi}_{t+1} = \\ &= \kappa x_t + u_t + \sum\nolimits_{j=1}^{\infty} \beta^j \left(\kappa x_{t+j} + u_{t+j} - \eta^{\pi}_{t+j}\right).\end{aligned}$$

Stability requires that $\kappa x_{t+j} + u_{t+j} - \eta^{\pi}_{t+j} = 0, \forall j$, so that the solution for the forecast error $\eta^{\pi}_t = \pi_t - E_{t-1}\pi_t = \kappa x_t + u_t$. Consequently, $\pi_t = \kappa x_t + u_t$. Since $E_t x_{t+1} = 0$, we can specify a general solution $x_t = \gamma_u u_t + \gamma_{\overline{y}}\overline{y}_t + \gamma_b b_t$, following Farmer, Khramov, and Nicolò (2015). $b_t \sim_{iid} N(0,1)$ is a normalized sunspot or belief shock, while γ_u, $\gamma_{\overline{y}}$, and γ_b are the respective loadings on the disturbances, which have to be consistent with a rational expectations equilibrium. The set of potential solutions to the simple NK model under imperfect information can thus be written as:

$$\begin{aligned}\pi_t &= (1 + \kappa\gamma_u)\, u_t + \kappa\gamma_{\overline{y}}\overline{y}_t + \kappa\gamma_b b_t, \\ y_t &= \gamma_u u_t + (1 + \gamma_{\overline{y}})\,\overline{y}_t + \gamma_b b_t, \\ x_t &= \gamma_u u_t + \gamma_{\overline{y}}\overline{y}_t + \gamma_b b_t.\end{aligned}$$

As the central bank's information set is nested inside the public's information set, the model's expectational linear-difference system can be conditioned down onto the central bank's information. As a result, any equilibrium has to be consistent with central bank projections, which we refer to as the projection condition. In the present example, the projection condition leads to $\pi_t = \pi_{t|t} = g_{\pi u} u_{t|t} + g_{\pi\bar{y}}\bar{y}_{t|t}$ and $x_t = x_{t|t} = g_{xu} u_{t|t} + g_{x\bar{y}}\bar{y}_{t|t}$, where $g_{\pi u}$, $g_{\pi\bar{y}}$, g_{xu}, $g_{x\bar{y}}$ are known from the full-information solutions $\pi_t = g_{\pi u} u_t + g_{\pi\bar{y}}\bar{y}_t$ and $x_t = g_{xu} u_t + g_{x\bar{y}}\bar{y}_t$. Specifically, the full-information coefficient values are:

$$\begin{aligned} g_{xu} &= -\frac{\phi_\pi}{\sigma + \kappa\phi + \phi_x}, & g_{\pi u} &= 1 + \kappa g_{xu}, \\ g_{x\bar{y}} &= -\frac{\sigma}{\sigma + \kappa\phi + \phi_x}, & g_{\pi\bar{y}} &= \kappa g_{x\bar{y}}, \end{aligned}$$

To derive the set of all admissible solutions, we find it useful to rewrite the imperfect-information solution in terms of deviations from the full-info case:

$$\begin{aligned} x_t &= g_{xu} u_t + g_{x\bar{y}}\bar{y}_t + \delta_t \\ \pi_t &= g_{\pi u} u_t + g_{\pi\bar{y}}\bar{y}_t + \kappa\delta_t \\ y_t &= g_{xu} u_t + (1 + g_{x\bar{y}})\bar{y}_t + \delta_t \end{aligned}$$

with

$$\delta_t = \tilde{\gamma}_u u_t + \tilde{\gamma}_{\bar{y}} \bar{y}_t + \gamma_b b_t \,,$$
$$\tilde{\gamma}_u \equiv \gamma_u - g_{xu} \,,$$
$$\tilde{\gamma}_{\bar{y}} \equiv \gamma_{\bar{y}} - g_{x\bar{y}} \,.$$

The projection condition (for all variables) then reduces to $\delta_{t|t} = 0$. The full-information solution corresponds to $\delta_t = 0$, which trivially fulfills the projection condition.

To satisfy the projection condition, $\delta_{t|t} = E(\delta_t|\pi^t y^t) = 0$, we require δ_t to be uncorrelated with π_t and y_t at all times and at all lags. Since δ_t, π_t and y_t are iid in this specific example, the following covariance conditions are necessary and sufficient:

$$\mathrm{Cov}\,(\delta_t, \pi_t) = 0 \,, \qquad \text{and} \qquad \mathrm{Cov}\,(\delta_t, y_t) = 0 \,. \tag{A-1}$$

By enforcing an orthogonality, that is, zero covariance condition, we can simplify the variance-covariance matrix of inflation and output in the Kalman gains without solving for a matrix inverse. In general, this is not possible and hinges on the specific assumptions for the information set. Given the $i.i.d.$ property of all variables involved, signal extraction involves the following Kalman gain, K:

$$\delta_{t|t} = K \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \,, \qquad \text{with} \quad K = \begin{bmatrix} \mathrm{Cov}\,(\delta_t, \pi_t) \\ \mathrm{Cov}\,(\delta_t, y_t) \end{bmatrix} \mathrm{Var}\left(\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \right)^{-1} . \tag{A-2}$$

The two equations of the projection condition (A-1) yield two equations in three unknowns:

$$\mathrm{Cov}\,(\delta_t, \pi_t) = \tilde{\gamma}_u \cdot (\kappa g_{xu} + 1)\, \sigma_u^2 + \tilde{\gamma}_{\bar{y}} \cdot (\kappa g_{x\bar{y}})\, \sigma_{\bar{y}}^2 + \kappa\, \mathrm{Var}\,(\delta_t) \qquad \stackrel{!}{=} 0 \,, \tag{A-3}$$
$$\mathrm{Cov}\,(\delta_t, y_t) = \tilde{\gamma}_u \cdot g_{xu} \sigma_u^2 + \tilde{\gamma}_{\bar{y}} \cdot (\kappa g_{x\bar{y}} + 1)\, \sigma_{\bar{y}}^2 + \mathrm{Var}\,(\delta_t) \qquad \stackrel{!}{=} 0 \,, \tag{A-4}$$

with

$$\mathrm{Var}\,(\delta_t) = \tilde{\gamma}_u^2 \sigma_u^2 + \tilde{\gamma}_{\bar{y}}^2 \sigma_{\bar{y}}^2 + \gamma_b^2 \,. \tag{A-5}$$

Equations (A-3) and (A-4) are two *non-linear* equations in the three unknowns $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, γ_b. Alternatively, we can consider $\mathrm{Var}\,(\delta_t)$ a yet to be determined unknown, so that equations (A-3) and (A-4) are two linear equations in the three unknowns $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, and $\mathrm{Var}\,(\delta_t)$. These equations are straightforward to solve and lead to expressions for $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$ as functions of parameters and $\mathrm{Var}\,(\delta_t)$.

With those solutions in hand, equation (A-5) becomes a quadratic equation in $\mathrm{Var}\,(\delta_t)$, with a single unknown parameter, γ_b^2. Intuitively, γ_b^2 reflects the extent to which belief shocks matter, while $\mathrm{Var}\, delta_t$ measures the amount of fluctuations in the output gap due to imperfect information. The quadratic equation for $\mathrm{Var}\,(\delta_t)$ has the form

$$a \cdot \mathrm{Var}\,(\delta_t)^2 - \mathrm{Var}\,(\delta_t) = -\gamma_b^2$$

with $a > 0$ reflecting the solution to (A-3) and (A-4). As noted in the main text above, we can then derive that

$$\gamma_b^2 < \frac{1}{4 \cdot a}\,, \quad \text{with} \quad a = \left[\frac{\kappa^2}{\sigma_u^2} + \frac{1}{\sigma_{\bar{y}}^2}\right] \cdot \left(\frac{\sigma + \kappa\phi_\pi + \phi_x}{\phi_x}\right)^2 . \tag{10}$$

To see this, let

$$\boldsymbol{v}_t = \begin{bmatrix} u_t \\ \bar{y}_t \end{bmatrix}, \qquad \boldsymbol{\gamma} = \begin{bmatrix} \tilde{\gamma}_u \\ \tilde{\gamma}_{\bar{y}} \end{bmatrix}, \qquad \boldsymbol{\psi} = \begin{bmatrix} \kappa \\ 1 \end{bmatrix}, \qquad \boldsymbol{G} = \begin{bmatrix} g_{\pi,u} & g_{\pi,\bar{y}} \\ g_{y,u} & g_{y,\bar{y}} \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_{\bar{y}}^2 \end{bmatrix},$$

so that we have

$$\delta_t = \boldsymbol{\gamma}'\boldsymbol{v}_t + \gamma_b b_t\,, \qquad\qquad \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \boldsymbol{G}\boldsymbol{v}_t + \begin{bmatrix} \kappa \\ 1 \end{bmatrix} \delta_t\,,$$

and the covariance restrictions emanating from the projection condition in (A-2) can be restated as

$$\mathrm{Cov}\left(\delta_t, \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}\right) = \boldsymbol{\gamma}'\boldsymbol{\Omega}\boldsymbol{G}' + \mathrm{Var}\,(\delta_t)\boldsymbol{\psi}\,.$$

To set the covariances between δ_t and inflation and output to zero, respectively, we obtain

$$\boldsymbol{\gamma}' = -\,\mathrm{Var}\,(\delta_t)\boldsymbol{\psi}'\left(\boldsymbol{G}^{-1}\right)'\boldsymbol{\Omega}^{-1}\,.$$

From $\mathrm{Var}\,(\delta_t) = \boldsymbol{\gamma}'\Omega\boldsymbol{\gamma} + \gamma_b^2$ we can deduce that $a = \boldsymbol{\alpha}'\boldsymbol{\alpha}$ with:

$$\boldsymbol{\alpha} = \boldsymbol{\Omega}^{-1/2}\boldsymbol{G}^{-1}\boldsymbol{\psi}$$

and straightforward algebra yields the result in (10).

A.2 Extension to a Taylor rule with interest-rate smoothing

As an example of letting policy respond also to past conditions, consider adding interest-rate smoothing to the Taylor rule. In the perfect-information case the rule can then be written as follows:

$$i_t = \phi_i i_{t-1} + (1-\phi_i)(\phi_\pi \pi_t + \phi_x x_t) = (1-\phi_i)\sum_{k=0}^{\infty} \phi_i^k (\phi_\pi \pi_{t-k} + \phi_x x_{t-k}) \qquad \text{(A-6)}$$

As shown above, interest-rate smoothing can also be viewed as monetary policy responding to lagged outcomes. Woodford (2003) offers a prominent discussion of likely benefits for policy design from such a dependency of policy rules on lagged policy rates, and similar rules are also discussed regularly in the Federal Reserve Board's Monetary Policy Report to the U.S. Congress.

As before, we maintain the standard assumption that inflation and output are perfectly observed by the policymaker, while the output gap is not directly observed. The policy maker has perfect recall, so that her information set reflects the entire history of current and past values of inflation and output, π_t and y_t. The imperfect-information version of the augmented Taylor rule in (A-6) then becomes:

$$i_t = \phi_i i_{t-1} + (1-\phi_i)(\phi_\pi \pi_t + \phi_x x_{t|t})$$

The projection condition requires the policymaker's optimal projections to be consistent with decision rules known from the full-information equilibrium, which in turn are consistent with equilibrium conditions evaluated at the policymaker's projections for states and outcomes. In particular, the policymaker's interest rate setting has to be consistent with the (projected) Fisher equation:[1]

$$i_t = r_{t|t} + \pi_{t+1|t} = \sigma(y_{t+1|t} - y_{t|t}) + \pi_{t+1|t}$$

With interest-rate smoothing, the lagged interest rate becomes a state variable (under full and imperfect information), and full-information decision rules take the form $\pi_t = g_{\pi,u} u_t + g_{\pi,\bar{y}} \bar{y}_t + g_{\pi,i} i_{t-1}$ (and similarly for other variables), with coefficients obtained via matrix methods, such as those in Klein (2000). Of note, the resulting values for $g_{\pi,u}$, $g_{y,u}$ etc. are different than those derived for the example without interest-rate smoothing in Appendix A.1.

With u_t and $\bar{y}_t$ being $i.i.d.$, and i_t known to the policymaker, it follows that $\pi_{t+1|t} = g_{\pi,i}\, i_t$. With

[1] The same logic applies in case of the example without interest-rate smoothing described in Appendix A.1. In the absence of endogenous state variables, $i.i.d.$ shocks and observable output, that example leads to $\pi_{t+1|t} = 0$ and $r_{t|t} = -\sigma y_{t|t} = -\sigma y_t = r_t$.

y_t directly observed as well, we can conclude that $y_{t+1|t} = g_{y,i}\, y_t$ and thus that

$$i_t = -\frac{\sigma}{1 - \sigma g_{y,i} - g_{\pi,i}} \cdot y_t \,. \tag{A-7}$$

The same result can be obtained by following the steps described for the case without interest rate smoothing in Appendix A.1:

a. Invert the full-information decision rules for inflation and output to back out the policymaker's projections $u_{t|t}$ and $\bar{y}_{t|t}$ as linear combination of inflation, output and lagged interest rate, which are all known to the policymaker.

b. Construct an output gap projection from the projection condition, $x_{t|t} = g_{x,u} u_{t|t} + g_{x,\bar{y}} \bar{y}_{t|t} + g_{x,i} i_{t-1}$.

c. Finally, when the output gap projection is plugged into the Taylor rule, we obtain (A-7) as effective Taylor rule that takes into account the effects of the projection condition.

An algebraic derivation via this alternative approach is complex, but the approach via an inversion of decision rules can be used to numerically verify the solution in (A-7).

An important upshot of (A-7) is that interest-rate smoothing does not undo the indeterminacy found earlier. Since the effective Taylor rule in (A-7) responds only to output, and not to inflation (nor to lagged policy rates), the resulting dynamic system falls in the class of simple outcome-based policy rules studied by Bullard and Mitra (2002), and fails their necessary and sufficient criterion for determinacy.[2]

Despite the apparent isomorphism in terms of evaluating the determinacy criterion of the effective Taylor rule in the augmented model with interest-rate smoothing, the resulting equilibria are richer than in the case of a simple Taylor rule. While the lagged interest rate plays no rule in checking for the effective Taylor rule's determinacy criterion, we still require that (projected) outcomes satisfy the projection condition as applied to the full-information outcomes of the augmented model, which feature a dependence of i_{t-1}. Specifically, equilibrium outcomes need to be characterized by

$$\begin{aligned} y_t &= g_{y,u}\, u_{t|t} + g_{y,\bar{y}}\, \bar{y}_{t|t} + g_{y,i}\, i_{t-1} \\ \pi_t &= g_{\pi,u}\, u_{t|t} + g_{\pi,\bar{y}}\, \bar{y}_{t|t} + g_{\pi,i}\, i_{t-1} \\ i_t &= g_{i,u}\, u_{t|t} + g_{i,\bar{y}}\, \bar{y}_{t|t} + g_{i,i}\, i_{t-1} \end{aligned}$$

[2] The condition from Bullard and Mitra (2002) for our case is $\kappa(\bar{\phi}_\pi - 1) + (1-\beta)]bar\phi_y > 0$. It is never satisfied by the effective Taylor rule in (A-7), which has response coefficients of $\bar{\phi}_\pi = 0$ and $\bar{\phi}_y < 0$.

where "$g_{\cdot,\cdot}$" are coefficients known from the full-information solution of the NK model with the interest-rate smoothing rule in (A-6).[3]

Based on our solution for the simple Taylor-rule model in Appendix A.1, the $i.i.d.$ nature of u_t and $\bar{y}_t$ and since the only endogenous state variable, i_{t-1} is observed, we deduce that equilibrium outcomes are characterized as the sum of their full-information outcomes, plus a univariate, $i.i.d.$ process, δ_t, that measures the effects of indeterminacy:

$$\begin{aligned}
\delta_t &= \tilde{\gamma}_u u_t + \tilde{\gamma}_{\bar{y}} \bar{y}_t + \gamma_b b_t \\
y_t &= \qquad \delta_t + g_{y,u}\, u_t + g_{y,\bar{y}}\, \bar{y}_t + g_{y,i}\, i_{t-1} \\
\pi_t &= \psi_\pi\, \delta_t + g_{\pi,u}\, u_t + g_{\pi,\bar{y}}\, \bar{y}_t + g_{\pi,i}\, i_{t-1} \\
i_t &= \psi_i\, \delta_t\ + g_{i,u}\, u_t + g_{i,\bar{y}}\, \bar{y}_t + g_{i,i}\, i_{t-1}
\end{aligned}$$

with

$$\begin{aligned}
& \psi_\pi = \kappa + \beta g_{\pi,i} \psi_i\,, && \psi_i = (1-\phi_i)(\phi_\pi \psi_\pi + \phi_x)\,, \\
\Leftrightarrow \qquad & \psi_\pi = \frac{\kappa + \beta g_{\pi,i}(1-\phi_i)\phi_x}{1-\beta g_{\pi,i}(1-\phi_i)\phi_\pi}\,, && \psi_i = (1-\phi_i)\frac{\phi_x + \kappa\,\phi_\pi}{1-\beta g_{\pi,i}(1-\phi_i)\phi_\pi}
\end{aligned}$$

The variable δ_t is expressed in units of output, and the coefficients ψ_i and ψ_π reflect the effects of δ_t on the policy rate and inflation.

As in case of the simple Taylor rule discussed in in Appendix A.1, the projection condition requires that $\delta_{t|t} = 0$ leading the covariance restrictions $\text{Cov}\,(\pi_t, \delta_t) = 0$ and $\text{Cov}\,(y_t, \delta_t) = 0$.[4] Proceeding analogously to the simple case, the covariance restrictions lead to one degree of freedom for the three parameters γ_u, $\gamma_{\bar{y}}$ and γ_b (albeit evaluated at coefficient values $g_{\cdot,\cdot}$ applicable in the model with interest rate smoothing).

The problem of solving for $\tilde{\gamma}_u$, $\tilde{\gamma}_{\bar{y}}$, and γ_b is isomorphic to the case without interest-rate smoothing described in Appendix A.1, but using the coefficients $g_{\pi,u}$, $g_{\pi,\bar{y}}$, $g_{y,u}$, etc. that characterize the full-information solution in the case of the augmented Taylor rule in (A-6), and with $\boldsymbol{\psi} = \begin{bmatrix}\psi_\pi & 1\end{bmatrix}'$. A crucial difference to the simple case is that through the dependence of policy rates on lagged rates, the contemporaneous effects of δ_t now lead to the propagation of belief shocks to current and future outcomes for inflation and output.

[3]The full-information decision rules of the NK model with the interest-rate smoothing rule in (A-6) differ, of course, from those derived in closed form for the case of a simple rule as derived in Appendix A.1. They reflect the solution to a second-order difference equation, which is straightforward to obtain using matrix methods such as in Blanchard and Kahn (1980), King and Watson (1998), Klein (2000), or Sims (2002).

[4]Note: The dependence of π_t and y_t on i_{t-1} has no effect on these covariance restrictions, since δ_t is iid.

A.3 NK Model with Observed Inflation Only

We assume that the central bank's information set is $Z_t = \{\pi_t\}$, which implies that $\pi_{t|t} = \pi_t$. Moreover, we assume that $\overline{y}_t \equiv 0$, so that there are no shocks to potential output and $r_t = 0$. Consequently, the central bank has to project $x_{t|t}$ and $u_{t|t}$. The unique rational expectations solution under full information is:

$$\pi_t = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_t\,, \qquad\qquad x_t = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_t.$$

Similarly, the projection condition is:

$$\pi_{t|t} = \frac{\phi_x + \sigma}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t}\,, \qquad\qquad x_{t|t} = -\frac{\phi_\pi}{\phi_x + \sigma + \kappa\phi_\pi} u_{t|t}.$$

A.3.1 Overview of Solution Approach

Since $\pi_{t|t} = \pi_t$, we can write $x_{t|t} = -\frac{\phi_\pi}{\phi_x+\sigma}\pi_t$ and use this to substitute out the projected output gap from the policy rule:

$$i_t = \frac{\sigma\phi_\pi}{\phi_x + \sigma}\pi_t = \overline{\phi}_\pi \pi_t.$$

and results in a liner-difference system for inflation and output gap of the following form:

$$\begin{bmatrix} \pi_{t+1} \\ x_{t+1} \end{bmatrix} = \frac{1}{\sigma\beta} \begin{bmatrix} \sigma & -\sigma\kappa \\ \beta\bar{\phi} - 1 & \kappa + \beta\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \boldsymbol{\eta}_{t+1} + \begin{bmatrix} -\frac{1}{\beta} \\ \frac{1}{\sigma\beta} \end{bmatrix} u_t, \tag{A-8}$$

The 'reduced-form' policy rule implies a unique equilibrium if $\overline{\phi}_\pi > 1$, or equivalently, $\phi_\pi > 1 + \frac{1}{\sigma}\phi_x$. Under the latter restriction, the solution to the imperfect information economy is identical to its full-information counterpart.

If $\phi_\pi < 1 + \frac{1}{\sigma}\phi_x$, the equilibrium of the imperfect information model is indeterminate. Mechanically, this implies that only one root of the dynamic system in (A-8) is unstable, while the other is stable, which results in endogenous persistence as discussed in Lubik and Schorfheide (2003). Closed-form solutions are hard to obtain for this system, but the remainder of this appendix describes a numerical solution.

A.3.2 Details of Solution Framework

The dynamic system in (A-8) has the following general format:

$$\boldsymbol{y}_{t+1} = \boldsymbol{B}\,\boldsymbol{y}_t + \boldsymbol{\eta}_{t+1} + \boldsymbol{D}\,\boldsymbol{v}_t \tag{A-9}$$

where $\boldsymbol{\eta}_t$ is a vector of yet-to-be-determined endogenous forecast errors, and $\boldsymbol{y}_t$ is a vector of jump variables. The projection condition is given by $\boldsymbol{y}_{t|t} = \boldsymbol{G}\ \boldsymbol{v}_{t|t}$ where $\boldsymbol{G}$ reflects the full-information solution.

MSV equilibrium: Given our assumptions about parameter values, the full information solution is determinate, and we can recover $\boldsymbol{G}$ also as solution to the minimal-state-variable (MSV) equilibrium to the system in (A-9). When all elements of $\boldsymbol{y}_{t+1}$ are jump variables, and provided that $|\boldsymbol{B}| \neq 0$, the MSV approach leads to

$$\boldsymbol{y}_t = -\boldsymbol{B}^{-1}\boldsymbol{D}\ \boldsymbol{v}_t\,, \qquad \text{with} \quad -\boldsymbol{B}^{-1}\boldsymbol{D} = \boldsymbol{G}\,, \tag{A-10}$$

and $\boldsymbol{G}$ collecting the coefficients of the full-information solution.[5] The reason is that the projection condition used to rewrite Taylor rule ensures that the MSV solutions under full information carries over also the system in (A-9). As a corollary, the MSV solution trivially satisfies the projection condition as well. The remainder of this section derives a set of additional REE that are stable, satisfy the projection condition and solve (A-9).

Canonical variables and other equilibria: Henceforth, we assume that the determinacy condition does not hold, so that $\boldsymbol{B}$ has one stable and one unstable root so that (A-10) is not unique unless the projection condition pins down $\boldsymbol{\eta}_t$ as a unique function of fundamental shocks. Moreover, since all roots have different moduli, it is assured that all are real.[6]

Consider the Schur decomposition of $\boldsymbol{B} = \boldsymbol{Z}'\boldsymbol{T}\boldsymbol{Z}$ where $\boldsymbol{T}$ is block upper triangular and $\boldsymbol{Z}$ is an orthonormal matrix, so that $\boldsymbol{Z}'\boldsymbol{Z} = \boldsymbol{I}$. Let the Schur decomposition be ordered, such that the stable roots of $\boldsymbol{B}$ come first and we have the following transformation into canonical variables:

$$\begin{aligned} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} &= \boldsymbol{Z}\boldsymbol{y}_t\,, \\ \Longrightarrow \quad \begin{bmatrix} \chi_{t+1} \\ \gamma_{t+1} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{0} & \boldsymbol{T}_{22} \end{bmatrix} \begin{bmatrix} \chi_t \\ \gamma_t \end{bmatrix} + \tilde{\boldsymbol{\eta}}_{t+1} + \boldsymbol{Z}\boldsymbol{D}\boldsymbol{v}_t \end{aligned} \tag{A-11}$$

where $\tilde{\boldsymbol{\eta}}_{t+1} = \boldsymbol{Z}\boldsymbol{\eta}_{t+1}$ (we will denote appropriate partitions of $\tilde{\boldsymbol{\eta}}_t$ by $\tilde{\boldsymbol{\eta}}_t^{\chi}$ and $\tilde{\boldsymbol{\eta}}_t^{\gamma}$). Crucially, γ_t is the unstable canonical variable (vector), since all roots of $\boldsymbol{T}_{22}$ are outside the unit circles; conversely, χ_t is the stable canonical variable. (In our specific example, γ_t, χ_t, $\boldsymbol{T}_{11}$, $\boldsymbol{T}_{12}$, and $\boldsymbol{T}_{22}$ are scalars.)

[5] If $|\boldsymbol{B}| = 0$, we can use the MSV solution of the original full-information economy, which still applies.

[6] Since $\boldsymbol{B}$ is real, complex eigenvalues would arise only in complex-conjugate pairs, each sharing the same modulus.

In terms of canonical variables, the benchmark equilibrium then implies

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\chi}_t \\ \boldsymbol{\gamma}_t\end{bmatrix} &= -\boldsymbol{T}^{-1}\boldsymbol{Z}\boldsymbol{D}\,\boldsymbol{v}_t = \begin{bmatrix}-\boldsymbol{T}_{11}^{-1}\tilde{\boldsymbol{D}}_\chi + \boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\boldsymbol{T}_{22}^{-1}\tilde{\boldsymbol{D}}_\gamma \\ -\boldsymbol{T}_{22}^{-1}\tilde{\boldsymbol{D}}_\gamma\end{bmatrix}\boldsymbol{v}_t \\ &= \begin{bmatrix}\tilde{\boldsymbol{G}}_\chi \\ \tilde{\boldsymbol{G}}_\gamma\end{bmatrix}\boldsymbol{v}_t\end{aligned}$$

with $\tilde{\boldsymbol{D}}_\chi$ and $\tilde{\boldsymbol{D}}_\gamma$ being appropriate partitions of $\tilde{\boldsymbol{D}} \equiv \boldsymbol{Z}\boldsymbol{D} = \begin{bmatrix}\tilde{\boldsymbol{D}}_\chi \\ \tilde{\boldsymbol{D}}_\gamma\end{bmatrix}$.

For any stable solution of (A-11), the unstable canonical variables must be solved forward, and we obtain the benchmark outcomes for $\boldsymbol{\gamma}_t$ that apply in any stable equilibrium:

$$\boldsymbol{\gamma}_t = \tilde{\boldsymbol{\eta}}_t^\gamma = \tilde{\boldsymbol{G}}_\gamma\,\boldsymbol{v}_t\,. \tag{A-12}$$

Translated into canonical variables, the projection condition requires that the projected canonical variables are equal to the projected benchmark outcomes:

$$\begin{bmatrix}\boldsymbol{\chi}_{t|t} \\ \boldsymbol{\gamma}_{t|t}\end{bmatrix} = \begin{bmatrix}\tilde{\boldsymbol{G}}_\chi \\ \tilde{\boldsymbol{G}}_\gamma\end{bmatrix}\boldsymbol{v}_{t|t} \tag{A-13}$$

and with (A-12) the projection condition *for the unstable variables* $\boldsymbol{\gamma}_t$ is trivially fulfilled *for any stable equilibrium.*

Stable canonical variables and projection condition: It remains to be seen whether the projection condition is sufficient to rule out indeterminate outcomes for the stable variables. It will be useful to focus on deviations of $\boldsymbol{\chi}_t$ from the benchmark outcomes, and see under which conditions the deviations are zero. Collecting terms we have:

$$\begin{aligned}\boldsymbol{\delta}_{t+1} &\equiv \boldsymbol{\chi}_{t+1} - \tilde{\boldsymbol{G}}_\chi\,\boldsymbol{v}_{t+1} = \boldsymbol{T}_{11}\boldsymbol{\delta}_t + \tilde{\boldsymbol{\eta}}_{t+1}^\delta\,, \\ \text{with}\quad \tilde{\boldsymbol{\eta}}_{t+1}^\delta &= \tilde{\boldsymbol{\eta}}_{t+1}^\chi - \tilde{\boldsymbol{G}}_\chi\boldsymbol{v}_t = \tilde{\boldsymbol{\Gamma}}_v\boldsymbol{v}_{t+1} + \tilde{\boldsymbol{\Gamma}}_b\boldsymbol{b}_{t+1}\end{aligned} \tag{A-14}$$

where $\boldsymbol{b}_t \sim N(0, I)$ is a (vector of) non-fundamental belief shock(s). $\tilde{\boldsymbol{\Gamma}}_v$ and $\tilde{\boldsymbol{\Gamma}}_b$ are (yet to be determined) shock loadings.

The central bank's filtering problem is characterized by the state equation (A-14) for $\boldsymbol{\delta}_t$, together with the following measurement equation:

$$\boldsymbol{z}_t = \boldsymbol{Z}^{z,\chi}\boldsymbol{\delta}_t + \boldsymbol{G}_z\boldsymbol{v}_t \tag{A-15}$$

where $\boldsymbol{z}_t = \pi_t$ in case of one example of Section 2, and $\boldsymbol{z}_t = \boldsymbol{y}_t$ in case of the other example of Section 2, and $\boldsymbol{Z}^{z,\chi}$ and $\boldsymbol{G}_z$ refer to the corresponding partitions of $\boldsymbol{Z}'$ and $\boldsymbol{G}$. In addition, let $\mathrm{Var}\,(\boldsymbol{v}_t) = \boldsymbol{\Omega}_v$.

We seek to find parameters for the shock loadings of $\boldsymbol{\delta}_t$, that is $\tilde{\boldsymbol{\Gamma}}_v$ and $\tilde{\boldsymbol{\Gamma}}_b$, that satisfy the projection condition. With (A-15) the projection reduces to the following:

$$\boldsymbol{\delta}_{t|t} = \boldsymbol{0} \quad \Leftrightarrow \quad \mathrm{Cov}\,(\boldsymbol{\delta}_t, \boldsymbol{z}_t | \boldsymbol{z}^{t-1}) = \mathrm{Cov}\,(\boldsymbol{\delta}_t, \boldsymbol{z}_t) = \boldsymbol{0}\,. \tag{A-16}$$

The conventional determinacy requirement is to solve unstable canonical variables forward. Our imperfect-information framework adds the condition that projections of stable canonical variables have to be equal to their projected solution in the benchmark (or MSV) equilibrium. To preview the derivation: Let $\boldsymbol{\Sigma} = \mathrm{Var}\,(\boldsymbol{\delta}_t|\boldsymbol{z}^t)$ and note that the projection condition (A-16) implies that $\mathrm{Var}\,(\boldsymbol{\delta}_t|\boldsymbol{z}^t) = \mathrm{Var}\,(\boldsymbol{\delta}_t)$. We then use the covariance restriction (A-16) to solve for $\tilde{\boldsymbol{\Gamma}}_v$ as a function of $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Gamma}}_b$ (and other fixed model parameters). Substituting out $\tilde{\boldsymbol{\Gamma}}_v$ yields a quadratic equation for $\boldsymbol{\Sigma}$ which merely places an upper bound on $\tilde{\boldsymbol{\Gamma}}_b$, but without uniquely determining the belief shock. In particular, the belief shock loadings need not be zero. In detail:

$$\begin{aligned} \mathrm{Cov}\,(\boldsymbol{\delta}_t, \boldsymbol{z}_t) = \boldsymbol{\Sigma}(\boldsymbol{Z}^{z,\chi})' + \tilde{\boldsymbol{\Gamma}}_v \boldsymbol{\Omega}_v \boldsymbol{G}_z' \qquad & \stackrel{!}{=} \boldsymbol{0} \\ \Rightarrow \tilde{\boldsymbol{\Gamma}}_v = -\boldsymbol{\Sigma}(\boldsymbol{Z}^{z,\chi})'(\boldsymbol{G}_z')^{-1}\boldsymbol{\Omega}_v^{-1} & \end{aligned} \tag{A-17}$$

provided that $\boldsymbol{\Omega}_v \boldsymbol{G}_z'$ is square and invertible.[7] In addition, with $\boldsymbol{\Sigma} = \mathrm{Var}\,(\boldsymbol{\delta}_t|\boldsymbol{z}^t) = \mathrm{Var}\,(\boldsymbol{\delta}_t)$ and (A-14) we have the following expression for $\boldsymbol{\Sigma}$, from which we can substitute out $\tilde{\boldsymbol{\Gamma}}_v$ based on (A-17):

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{T}_{11}\boldsymbol{\Sigma}\,\boldsymbol{T}_{11}' + \tilde{\boldsymbol{\Gamma}}_v\boldsymbol{\Omega}_v\tilde{\boldsymbol{\Gamma}}_v' + \tilde{\boldsymbol{\Gamma}}_b\tilde{\boldsymbol{\Gamma}}_b' = \boldsymbol{T}_{11}\boldsymbol{\Sigma}\,\boldsymbol{T}_{11}' + \boldsymbol{\Sigma}\boldsymbol{\Psi}\boldsymbol{\Sigma} + \tilde{\boldsymbol{\Gamma}}_b\tilde{\boldsymbol{\Gamma}}_b' \\ \text{with} \quad \boldsymbol{\Psi} &= (\boldsymbol{Z}^{z,\chi})'(\boldsymbol{G}_z')^{-1}\boldsymbol{\Omega}_v^{-1}\boldsymbol{G}_z^{-1}\boldsymbol{Z}^{z,\chi} \end{aligned} \tag{A-18}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{T}_{11}$ are known and we seek to characterize the set of all $\tilde{\boldsymbol{\Gamma}}_b$ that admit a finite, symmetric and positive definite solution to $\boldsymbol{\Sigma}$: One solution that always exists, is $\tilde{\boldsymbol{\Gamma}}_b = \boldsymbol{0}$ which recovers the MSV solution (with $\boldsymbol{\Sigma} = \boldsymbol{0}$, $\tilde{\boldsymbol{\Gamma}}_v = \boldsymbol{0}$ and $\boldsymbol{\delta}_t = \boldsymbol{0}$). However, with $\boldsymbol{\Psi} \neq \boldsymbol{0}$, we have to worry about non-existence of an appropriate solution of $\boldsymbol{\Sigma}$ for arbitrary $\tilde{\boldsymbol{\Gamma}}_b$.

Putting all together In case of indeterminacy, outcomes are additive in two components, the MSV solution and the process for $\boldsymbol{\delta}$. Added persistence arising from indeterminacy is thus characterized

[7] Since $\boldsymbol{\Omega}_v$ is square and invertible, $\boldsymbol{\Omega}_v \boldsymbol{G}_z'$ being square and invertible requires $\boldsymbol{G}_z$ to be square and invertible, which in turn implies that there are as many fundamentals drivers (column of $\boldsymbol{G}_z$) as observables to the central bank (rows of $\boldsymbol{G}_z$).

by the stable root of the "effective" Taylor rule system, captured by $\boldsymbol{T}_{11}$.

$$\boldsymbol{\chi}_t = \tilde{\boldsymbol{G}}_\chi \boldsymbol{v}_t + \boldsymbol{\delta}_t\,, \qquad \boldsymbol{\gamma}_t = \tilde{\boldsymbol{G}}_\gamma \boldsymbol{v}_t\,, \qquad \Longrightarrow \quad \boldsymbol{y}_t = \boldsymbol{Z}' \begin{bmatrix} \boldsymbol{\chi}_t \\ \boldsymbol{\gamma}_t \end{bmatrix} = \boldsymbol{G}\boldsymbol{v}_t + \boldsymbol{Z}^\chi \boldsymbol{\delta}_t$$

where $\boldsymbol{Z}^\chi$ corresponds to the first block of columns of $\boldsymbol{Z}'$ that is associated with $\boldsymbol{\chi}_t$.[8]

Analytical solution in case of the earlier examples: In our examples, we have only one stable root of $\boldsymbol{B}$, so that $\boldsymbol{\chi}_t$ is a scalar, and so are $\boldsymbol{\delta}$, $\boldsymbol{T}_{11}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Sigma}$, and $\tilde{\boldsymbol{\Gamma}}_b$, which allows us to simplify further, and obtain:

$$a \cdot \boldsymbol{\Sigma}^2 - b \cdot \boldsymbol{\Sigma} = -\tilde{\boldsymbol{\Gamma}}_b^2 \qquad \text{with} \quad a = \boldsymbol{\Psi} > 0 \qquad \text{and} \quad b = 1 - \boldsymbol{T}_{11}^2 > 0\,.$$

which has the following non-negative solution, provided $\tilde{\boldsymbol{\Gamma}}_b^2 < 0.25 \cdot b^2/a$:

$$\Rightarrow \quad \boldsymbol{\Sigma} = \frac{1}{2}\frac{b}{a} \pm \sqrt{\left(\frac{1}{4}\frac{b^2}{a^2}\right) - \frac{\tilde{\boldsymbol{\Gamma}}_b^2}{a}}$$

A.4 A Fisher Economy with Measurement Error

A.4.1 Equilibrium with an Endogenous Signal

This section presents a formal statement of the equilibrium in the Fisher economy with a noisy inflation signal and serially correlated shocks. Exposition in the main text and in the additional material presented below assumes that the shocks are $i.i.d.$ for tractability. We assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. We present the analysis in terms of the projection equation for the real rate :

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t\right).$$

This leads to the full equation system:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\ r_{t|t} &= (\rho + \kappa_r)\, r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t. \end{aligned} \tag{A-19}$$

[8] Recall that $\tilde{\boldsymbol{G}}_\chi$ and $\tilde{\boldsymbol{G}}_\gamma$ are the appropriate partitions of $\tilde{\boldsymbol{G}} = \boldsymbol{Z}\boldsymbol{G}$.

The key coefficient is $(\rho + \kappa_r)$ on the lagged real rate projection, which determines the stability properties of the system but is not sufficient to determine equilibrium in contrast with the full information case. In addition, real rate projections depend on the endogenous forecast error η_t. As a result, the solution for κ_r depends on the equilibrium law of motion for π_t.

To solve the model, we first derive the endogenous Kalman gain and the associated forecast error variance. We then derive the projection condition and assess consistency with the proposed equilibrium paths. The steady-state Kalman gain is $\kappa_r = cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) \diagup var(\widetilde{Z}_t)$, where $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{\pi}_t + \nu_t$. As before, we decompose the endogenous forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t$. It can be quickly verified that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = -\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2$. The negative sign in this expression reflects the inverse relationship between inflation and the real rate when the signal is endogenous. Using $\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t$, we find that $var(\widetilde{Z}_t)$ can be expressed as $var(\widetilde{Z}_t) = \Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2$.

We can now derive the following expression for the Kalman gain:

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}. \tag{A-20}$$

Although the forecast error variance Σ still needs to be determined as a function of the structural parameters, we can make two observations already. First, the gain κ_r can be negative for small enough γ_ε, that is, $\kappa_r < 0$ if $\gamma_\varepsilon < \rho\Sigma/\sigma_\varepsilon^2$. Second, existence of a steady-state Kalman filter implies that $|\rho + \kappa_r| < 1$ as long as $\Sigma > 0$, as shown in Proposition 1 below. We return to a discussion of the case where $r_t = r_{t|t}$, so that $\Sigma = 0$, later in this section. In the next step, we compute the projection error variance $\Sigma = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$. Using $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ we can derive the following Riccati equation, which is quadratic in Σ:

$$\Sigma = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{\left(-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2\right)^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}. \tag{A-21}$$

Finally, an equilibrium has to obey the restrictions imposed by central bank projections, namely $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$. This implies a covariance restriction of projection errors which differs from the exogenous signal case because of different information sets. Specifically, we have that $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ or alternatively $(\phi - \rho)\, cov\left(\widetilde{\pi}_t, \widetilde{\pi}_t + \nu_t\right) = cov\left(\widetilde{r}_t, \widetilde{\pi}_t + \nu_t\right)$. After some rearranging we can write this expression as:

$$\gamma_\nu (1 + \gamma_\nu) = -\frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} - \frac{\gamma_b^2}{\phi - \rho} \frac{\sigma_b^2}{\sigma_\nu^2} + \frac{\left[1 - (\phi - \rho)\gamma_\varepsilon\right]\gamma_\varepsilon}{\phi - \rho} \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{A-22}$$

The condition places a quadratic restriction on all three innovation loadings γ. This can imply that there are no or multiple solution to this equation, and thus for the overall equilibrium, for a given parameterization of the model. We summarize our findings in the following proposition.

PROPOSITION 1 (Limited Information REE in the Fisher Economy) *The set of stationary rational expectations equilibria (REE) in the model (A-19) under LIRE with signal $Z_t = \pi_t + \nu_t$ is characterized by the following dynamic equations:*

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_\nu \nu_t + \gamma_b b_t, && \text{(A-23)}\\
r_{t|t} &= (\rho + \kappa_r)\, r_{t-1|t-1} - \kappa_r r_{t-1} + \kappa_r \gamma_\varepsilon \varepsilon_t + \kappa_r (1 + \gamma_\nu)\, \nu_t + \kappa_r \gamma_b b_t, && \text{(A-24)}\\
r_t &= \rho r_{t-1} + \varepsilon_t,
\end{aligned}$$

With $\Sigma > 0$, we have:

$$|\rho + \kappa_r| < 1 \tag{A-25}$$

$$\Sigma = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + 4\beta}\right), \tag{A-26}$$

$$\alpha = (1 + 2\rho\gamma_\varepsilon)\, \sigma_\varepsilon^2 - (1 - \rho)^2 \left(\gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2\right), \tag{A-27}$$

$$\beta = \left(\gamma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2\right) \sigma_\varepsilon^2, \tag{A-28}$$

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 + (1 + \gamma_\nu)^2 \sigma_\nu^2}, \tag{A-20}$$

$$\gamma_\nu (1 + \gamma_\nu) = -\frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} - \frac{1}{\phi - \rho} \frac{\sigma_b^2}{\sigma_\nu^2} + \left(\frac{1}{\phi - \rho} - \gamma_\varepsilon\right) \gamma_\varepsilon \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{A-22}$$

Proof. *Equations* (A-26)*,* (A-27) *and* (A-28) *follow directly from solving the quadratic equation for $\Sigma > 0$ in* (A-21)*. The expression for the Kalman gain κ_r in* (A-20) *and the restrictions from the projection condition in* (A-22) *restate earlier results. The requirement that with $\Sigma > 0$ we must have $|\rho + \kappa_r| < 1$ is an application of Theorem 2 in Appendix B.4. In this specific example, the result that $|\rho + \kappa_r| < 1$ can be derived as follows: Consider the candidate value $\kappa_r = 0$ for the Kalman gain; in this case we would have $\Sigma = \text{Var}(r_t)$. The optimal Kalman gain seeks to minimize Σ and the optimal value of Σ must thus be (weakly) smaller than $\text{Var}(r_t) = \sigma_\varepsilon^2/(1 - \rho^2)$ and finite. With the optimal Kalman gain, projections are given by* (A-24) *and the process for the projection errors $r_t^* = r_t - r_{t|t}$ is $r_t^* = (\rho + \kappa_r) r_{t-1}^* + \varepsilon_t - \kappa_r(\eta_t + \nu_t)$. Recall that $\Sigma \equiv \text{Var}(r_t^*)$. We can thus conclude that for $0 < \Sigma < \infty$ the optimal Kalman gain must be such that $|\rho + \kappa_r| < 1$.* ■

The Proposition describes the set of solutions under indeterminacy. With $|\rho + \kappa_r| < 1$ the equation system has only stable roots and therefore lacks a restriction to determine the endogenous forecast error uniquely. As in the case of an exogenous signal, the projection condition that ensures internal consistency of central bank and private sector expectation formation restricts the set of possible equilibria. Specifically, an equilibrium with $\Sigma > 0$ does not exist when no innovation loadings can be found to ensure existence of a steady-state Kalman filter that is consistent with the projection condition. Moreover, the set of solutions is restricted over the parameter space by the nonlinear Riccati equation for the forecast error variance, by non-negativity constraints on variances and by ruling out complex solutions.

In our framework, feedback between filtering and model solution is central to equilibrium determination. Filtering depends on the information set, the result of which affects equilibrium outcomes and the content of the information set. This fixed-point problem has been noted before, at least as early as Sargent (1991). We go beyond this insight by showing that equilibrium determination is substantially different from the standard linear RE case. A solution may not exist even when the root-counting criterion for existence of an equilibrium indicates a sufficient number of stable roots in standard full information settings. While the root-counting approach for given κ_r could indicate non-existence, uniqueness or indeterminacy, it is the second-moment restrictions due to the less informed agent's filtering problem that determine equilibrium. In that sense, indeterminacy is generic in this environment since existence of a stable Kalman filter introduces a stable root into the dynamic system. At the same time, the second-moment restrictions resulting from the projection condition restrain belief shock loadings, which stands in stark contrast to the case of indeterminacy in a full-information scenario.

A.4.2 Indeterminacy in the Fisher Example

We show that $|\phi\kappa_\pi| < 1$ in the simple example of Section 2 with $i.i.d.$ real rate shocks. The projection condition requires $\phi\kappa_\pi = \kappa_r$, where κ_r is the Kalman gain in the central bank's signal extraction for the real rate, $r_{t|t} = \kappa_r\left(\pi_t + \nu_t\right)$. Optimal signal extraction implies $|\kappa_r| < 1$ for any equilibrium where r_t cannot be perfectly inferred by the central bank.[9]

In the $i.i.d.$ case we have $\rho = 0$ and $r_t = \varepsilon_t$, and the central bank's real-rate projections are given by $r_{t|t} = \kappa_r\left(\pi_t + \nu_t\right)$ with $\kappa_r = \operatorname{Cov}\left(r_t, \pi_t + \nu_t \,|\, Z^{t-1}\right) / \operatorname{Var}\left(\pi_t + \nu_t \,|\, Z^{t-1}\right)$. Furthermore, recall the equilibrium process for inflation:

$$\pi_{t+1} = -\left(r_t - r_{t|t}\right) + \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1} \tag{A-29}$$

We now show that $|\kappa_r| < 1$ for any γ_ε, γ_ν, and γ_b. This carries over to $|\phi\kappa_\pi| < 1$ when the

[9]When r_t cannot be perfectly inferred by the central bank we have $\operatorname{Var}\left(r_{t|t}\right) < \operatorname{Var}\left(r_t\right)$.

projection condition is satisfied so that $\phi\kappa_\pi = \kappa_r$. Define the following variance ratio:

$$R_\varepsilon^2 \equiv \frac{\text{Var}\left(\varepsilon_{t|t}\right)}{\text{Var}\left(\varepsilon_t\right)} \qquad \text{with} \quad 0 \leq R_\varepsilon^2 \leq 1. \tag{A-30}$$

$$= \frac{\text{Cov}\left(\varepsilon_t, \pi_t + \nu_t\right)^2}{\text{Var}\left(\pi_t + \nu_t\right) \cdot \text{Var}\left(\varepsilon_t\right)}$$

$$= \frac{\gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2}{\text{Var}\left(\pi_t + \nu_t\right)} = \gamma_\varepsilon \cdot \kappa_r, \tag{A-31}$$

where the second line follows from optimal signal extraction and the last line from (A-29). Recall further that $r_t = \varepsilon_t$ in the iid case. Based on (A-29) and (A-30) we also have

$$\text{Var}\left(\pi_t + \nu_t\right) = \left(1 - R_\varepsilon^2\right) \cdot \sigma_\varepsilon^2 + \gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2 + (1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2$$

and (A-31) can be rewritten as

$$R_\varepsilon^2 \cdot \left(\left(1 - R_\varepsilon^2\right) \cdot \sigma_\varepsilon^2 + (1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2\right) = \left(1 - R_\varepsilon^2\right) \cdot \left(\gamma_\varepsilon^2 \cdot \sigma_\varepsilon^2\right) \tag{A-32}$$

First, consider the case of $R_\varepsilon^2 < 1$ so that the real rate is not perfectly revealed in equilibrium. Equation (A-32) then implies the following:

$$\gamma_\varepsilon^2 = R_\varepsilon^2 \cdot \left(1 + \frac{(1 + \gamma_\nu)^2 \cdot \sigma_\nu^2 + \gamma_b^2}{(1 - R_\varepsilon^2) \cdot \sigma_\varepsilon^2}\right) \geq R_\varepsilon^2.$$

With (A-31) we have $\kappa_r^2 = R_\varepsilon^2 \cdot (R_\varepsilon^2/\gamma_\varepsilon^2) < 1$ and thus $|\kappa_r| < 1$. The inequality is strict, since $R_\varepsilon^2 < 1$ in this case.

A.4.3 A Full Revelation Equilibrium

In the current example we can also construct an equilibrium with $|\kappa_r| = |\phi| > 1$ and where $R_\varepsilon^2 = 1$. In this case, (A-32) requires $\gamma_\nu = -1$ and $\gamma_b = 0$, from which it follows that $\pi_t = \bar{g} \cdot r_t - \nu_t$ with $\bar{g} = 1/\phi$ and thus $\kappa_r = \phi$. This equilibrium generates an unstable root in the linear RE system characterizing the economy since we have $\kappa_\pi = 1$ and thus $|\phi \cdot \kappa_\pi| > 1$. As before the unstable root reflects the endogenous Kalman gain.

Although this equilibrium shares features of the unique equilibrium under full information, we classify it as a belonging to the set of multiple equilibria under indeterminacy. In that sense it is similar to the continuity solution under indeterminacy as in Lubik and Schorfheide (2003), which is classified as a sunspot equilibrium without sunspots, where agents coordinate their beliefs on behavior that does not differ from its determinate counterpart. Nevertheless, it is just one of many equilibria that agents can coordinate on.

This equilibrium is identical in outcomes to what is generated by the policy rule described in (A-42). The construction of this alternative equilibrium straightforwardly applies also in the case where the real rate is not $i.i.d.$ but follows an AR(1) process.

A.4.4 An Alternative Derivation

We can derive existence and determinacy conditions in an alternative manner, using stability properties of lag polynomials. [10]

We consider the following system:

$$i_t = \phi \pi_{t|t} = \mathbb{E}_t \left[\pi_{t+1} \right] + r_t$$
$$\pi_{t|t} = \left(1 - \kappa_\pi\right) \rho \pi_{t-1|t-1} + \kappa_\pi \left(\pi_t + \nu_t\right).$$

The central bank's projection of inflation can be expressed as:

$$\pi_{t|t} = \frac{1}{1 - \left(1 - \kappa_\pi\right) \rho L} \kappa_\pi \left(\pi_t + \nu_t\right),$$

which then leads to:

$$\phi \frac{1}{1 - \left(1 - \kappa_\pi\right) \rho L} \kappa_\pi \left(\pi_t + \nu_t\right) = \mathbb{E}_t \left[\pi_{t+1} \right] + r_t$$

We can now state the dynamics for inflation directly as a moving average process (in contrast to the exposition in the rest of the paper):

$$\pi_t = g(L)\epsilon_t + h(L)\nu_t + d(L)b_t, \tag{A-33}$$

where $d(L)b_t$ denotes the dynamics driven by the sunspot shock b_t. Using the Hansen-Sargent prediction formula and the method of undetermined coefficients, it can be shown that:

$$\begin{aligned} g(L) &= \frac{-\frac{L(1-(1-\kappa_\pi)\rho L)}{1-\rho L} + g_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi + (1-\kappa_\pi)\rho)L} \\ h(L) &= \frac{\phi\kappa_\pi L + h_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi + (1-\kappa_\pi)\rho)L} \\ d(L) &= \frac{d_0(1-(1-\kappa_\pi)\rho L)}{1-(\phi\kappa_\pi + (1-\kappa_\pi)\rho)L} \end{aligned}$$

A unique equilibrium arises if $\left|\phi\kappa_\pi + \left(1 - \kappa_\pi\right)\rho\right| > 1$. This obtains because in this case $g(L), h(L)$ and $d(L)$ contain an explosive root, so that one can set g_0 and h_0 akin to solving the unstable equation forward. This leads to $d_0 = 0$ and the sunspot component is eliminated. On the other hand, if $\left|\phi\kappa_\pi + \left(1 - \kappa_\pi\right)\rho\right| < 1$, then there is no restriction on g_0, h_0 and d_0 (except that they

[10] We thank a referee for proposing this derivation of the solution for the Fisher model with persistent shocks.

have to be consistent with the Kalman filter), and hence we have multiple equilibria.

A.4.5 Closed-Form Solutions with An Alternative Information Set

We present a variant of the Fisher economy with a particular information set that enables us to derive a number of results in closed form.The model combines an exogenous AR(1) process for the real rate with a Fisher equation and a Taylor rule that responds to the central bank's inflation projection:

$$
\begin{aligned}
&i_t = r_t + E_t\pi_{t+1}\,, && i_t = \phi\,\pi_{t|t}\,, && |\phi| > 1\,,\\
&r_t = \rho\, r_{t-1} + \varepsilon_t && \varepsilon_t \sim iidN(0,\sigma_\varepsilon^2)\,, && |\rho| < 1\,,
\end{aligned}
$$

and the projection condition requires $\pi_{t|t} = r_{t|t}/(\phi-\rho)$. As before, we express the endogenous forecast error as a linear combination of fundamental and belief shocks. Collecting terms yields the following characterization of the inflation process:

$$\pi_{t+1} = -(r_t - r_{t|t}) + \frac{\rho}{\phi-\rho} r_t + \eta_{t+1} \tag{A-34}$$

$$
\begin{aligned}
\text{with} \quad \eta_{t+1} &\equiv \pi_{t+1} - E_t\pi_{t+1}\\
&= \gamma_\varepsilon \varepsilon_{t+1} + \gamma_\nu \nu_{t+1} + \gamma_b b_{t+1}\,, \qquad b_{t+1} \sim iidN(0,1)\,,
\end{aligned}
\tag{A-35}
$$

We now assume that the central bank's information set is characterized by a bivariate signal, which includes a perfect reading of the real rate and a noisy signal of current inflation:

$$\boldsymbol{Z}_t = \begin{bmatrix} r_t \\ \pi_t + \nu_t \end{bmatrix} \qquad \text{with} \quad \nu_t \sim iidN(0,\sigma_\nu^2) \tag{A-36}$$

$$\Rightarrow \quad r_{t|t} = r_t \quad \Rightarrow \quad \pi_{t|t} = \frac{1}{\phi-\rho} r_t\,, \tag{A-37}$$

In light of (A-37), the inflation dynamics specified in (A-34) simplify to

$$\pi_{t+1} = \frac{\rho}{\phi-\rho} r_t + \eta_{t+1}\,. \tag{A-38}$$

Before determining the shock loadings γ_ε, γ_ν, and γ_b of the endogenous forecast error η_t, we can already note that one-step-ahead expectations of inflation, $E_t\pi_{t+1} = r_t/(\phi-\rho)$, are identical to the full-information case so that the effects of indeterminacy will be limited to changes in the amplification of shocks, without consequences for inflation persistence.

In light of (A-38), we can conclude that the history of $\boldsymbol{Z}_t$ spans the same information content as the history of

$$\boldsymbol{W}_t = \begin{bmatrix} \varepsilon_t \\ (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t \end{bmatrix}.$$

$\boldsymbol{W}^t$ spans $\boldsymbol{Z}^t$ since, with $|\rho| < 1$, ε^t spans r^t, and since observing $(1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t$ adds the same information to the span of ε^t as observing $\pi_t + \nu_t$. The signal vector $\boldsymbol{W}_t$ consists of two mutually orthogonal elements that are serially uncorrelated over time. As a result, projections onto $\boldsymbol{W}^t$ can be decomposed into the sum of projections onto its individual elements.

The projection condition (A-37) then requires $\eta_{t|t} = \varepsilon_{t|t}/(\phi - \rho)$, and with $\varepsilon_{t|t} = \varepsilon_t$, we can conclude that

$$\gamma_\varepsilon = \frac{1}{\phi - \rho}. \tag{A-39}$$

In addition we have $\pi_t^* \equiv \pi_t - \pi_{t|t} = \gamma_\nu\,\nu_t + \gamma_b\, b_t$, and thus $E\left(\gamma_\nu\,\nu_t + \gamma_b\, b_t \,\middle|\, \boldsymbol{Z}^t\right) = 0$, which implies the following restriction on γ_ν and γ_b:

$$\mathrm{Cov}\left(\gamma_\nu\,\nu_t + \gamma_b\, b_t \,\middle|\, (1+\gamma_\nu)\,\nu_t + \gamma_b\, b_t\right) = \gamma_\nu(1+\gamma_\nu)\sigma_\nu^2 + \gamma_b^2 = 0\,. \tag{A-40}$$

$$\Rightarrow \quad \gamma_\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\gamma_b^2}{\sigma_\nu^2}} \tag{A-41}$$

Real solutions to (A-41) require $|\gamma_b| < 0.5\,\sigma_\nu$ leading to a continuum of solutions with $\gamma_\nu \in [-1, 0]$. While there is a unique solution for the shock loading γ_ε, as given in (A-39), there are multiple solutions for γ_ν and γ_b as characterized by (A-41).

Equilibrium dynamics of inflation are described by (A-34) and (A-35) together with (A-39) and (A-41). Collecting terms, inflation evolves according to:

$$\pi_t = \frac{1}{\phi - \rho} r_t + \gamma_\nu\,\nu_t + \gamma_b\, b_t$$

where γ_ν and γ_b are restricted by (A-41). In light of the projection condition (A-40), the variance of inflation is given by

$$\mathrm{Var}\,(\pi_t) = \left(\frac{1}{\phi - \rho}\right)^2 \mathrm{Var}\,(r_t) + |\gamma_\nu|\sigma_\nu^2\,.$$

Since γ_ν is bounded by one in absolute value, there is an upper bound for the inflation variance equal to $\mathrm{Var}\,(r_t)/(\phi-\rho)^2 + \sigma_\nu^2$.

A key feature of this example is that belief shock loadings are not generally zero, and non-fundamental belief shocks can affect equilibrium outcomes. In addition, there is an upper bound on belief shock loadings, which stems from the projection condition and is unique to our imperfect information framework. When indeterminacy arises in full-information models, there are no such bounds on the scale with which belief shocks can affect economic outcomes.

In the absence of measurement error on inflation, $\sigma_\nu = 0$, the outcomes in our simplified example collapse to the full-information solution $\pi_t = r_t/(\phi - \rho)$ and equilibria are continuous with respect to the full-information case as σ_ν approaches zero. For any measurement error variance, the range of possible equilibria includes the case where outcomes are identical to the full-information case, with $\gamma_\nu = \gamma_b = 0$.

A.4.6 Determinacy without Optimal Projections

In the model with imperfect information indeterminacy can arise from the interplay between the effects of policy on inflation and the policymaker's use of optimal projections. An alternative policy would be for the policymaker to respond directly to the noisy signal Z_t, rather than an optimal projection based on the signal. This alternative setup obviates the role of the endogenous Kalman gain κ_π. Consider the policy rule:

$$i_t = \phi\left(\pi_t + \nu_t\right) . \tag{A-42}$$

When $|\phi| > 1$, policy responds to actual inflation by more than one-for-one; the Taylor principle applies and results in a unique equilibrium, given by $\pi_t = \bar{g}\, r_t - \nu_t$. While the equilibrium is unique, inflation is subject to additional fluctuations, at least compared to the full-information case, which should be undesirable from the perspective of the policymaker. This is due to the presence of the measurement error ν_t. In fact, the projection condition guarantees that any of the multiple equilibria obtained under the projection-based Taylor rule imply a (weakly) lower variance of inflation than under the determinate policy rule in (A-42):

$$\mathrm{Var}\left(\pi_t\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_t\right) + \mathrm{Var}\left(\nu_t\right) \tag{A-43}$$

The projection condition implies $\mathrm{Var}\left(\pi_{t|t}\right) = \bar{g}^2\ \mathrm{Var}\left(r_t\right)$, and the optimality of projections leads to $\mathrm{Var}\left(\pi_{t|t}\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_{t|t}\right)$. Moreover, the definition of the signal implies $\pi_t - \pi_{t|t} = -(\nu_t - \nu_{t|t})$, since $Z_{t|t} = Z_t$, so that $\mathrm{Var}\left(\pi_t - \pi_{t|t}\right) = \mathrm{Var}\left(\nu_t - \nu_{t|t}\right)$. We obtain $\mathrm{Var}\left(\pi_t\right) = \mathrm{Var}\left(\pi_{t|t}\right) + \mathrm{Var}\left(\pi_t - \pi_{t|t}\right) \leq \bar{g}^2\ \mathrm{Var}\left(r_t\right) + \mathrm{Var}\left(\nu_t\right)$. Furthermore, recall that $\mathrm{Var}\left(\nu_t\right) = \sigma_\nu^2$ and, in the iid case, $\mathrm{Var}\left(r_t\right) = \sigma_\varepsilon^2$.

B General Framework

In this section we discuss the general framework laid out in Section 3 in more detail.

B.1 Full Information Equilibrium in the General Case

The general framework nests the case of full information, where $\boldsymbol{S}_{t+h|t} = E_t \boldsymbol{S}_{t+h} \; \forall h \geq 0$. This holds, for example, when $\boldsymbol{H} = \boldsymbol{I}$ such that $\boldsymbol{Z}_t = \boldsymbol{S}_t$. The full-information system can easily be solved using standards methods such as those found in King and Watson (1998), Klein (2000) or Sims (2002). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$.[11] We begin by restating some equations from the main part of the paper:

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix}, \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_t \end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix} \boldsymbol{Y}_t \\ \boldsymbol{i}_t \end{bmatrix}. \tag{16}$$

Using $\boldsymbol{\mathcal{S}}_t' = \begin{bmatrix} \boldsymbol{i}_{t-1}' & \boldsymbol{S}_t' & \boldsymbol{i}_t' \end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} + \hat{\boldsymbol{J}} & 0 \\ 0 & \boldsymbol{\Phi}_J & 0 \end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t \boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix} 0 & 0 & \boldsymbol{I} \\ 0 & \boldsymbol{A} + \hat{\boldsymbol{A}} & \boldsymbol{A}_i \\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathcal{S}}_t \tag{17}$$

Existence and uniqueness of a solution to this system depend on the properties of the matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. Throughout this paper we focus on environments where a unique full-information solution exists and therefore impose the following assumption known from Klein (2000) and King and Watson (1998).[12]

ASSUMPTION 1 (Unique full-information solution) *The set of generalized eigenvalues of the system matrices $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ as defined in* (17) *has $N_i + N_x$ roots inside the unit circle and $N_y + N_i$ roots outside the unit circle. Moreover, there is some complex number z such that $|\boldsymbol{\mathcal{J}} z - \boldsymbol{\mathcal{A}}| \neq 0$.*

The solution can, for instance, be computed using the numerical methods of Klein (2000), and

[11] The presence of the lagged policy control among the backward-looking variables in $\boldsymbol{\mathcal{X}}_t$ serves to handle the case of interest-rate smoothing, $\boldsymbol{\Phi}_i \neq 0$, and can otherwise be omitted.

[12] This is akin to a root-counting criterion where the number of explosive eigenvalues matches the number of forward-looking variables. Sims (2002) offers a slight generalization of this criterion. In the case of simple monetary policy models, the root-counting condition is satisfied by requiring that the central bank's interest-rate rule satisfies the Taylor principle, that is, by responding more than one-to-one to fluctuations in inflation.

has the following form:

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}}\ \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}}\ \boldsymbol{\mathcal{X}}_t\,, \qquad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}, \tag{18}$$

where $\boldsymbol{\mathcal{P}}$ is a stable matrix with all eigenvalues inside the unit circle. In the full-information version of our framework certainty equivalence holds; that is, the decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ nor on the measurement loadings $\boldsymbol{H}$.[13] Equilibrium dynamics in the full-information case are then summarized by:

$$\boldsymbol{\mathcal{S}}_{t+1} = \bar{\boldsymbol{\mathcal{T}}}\boldsymbol{\mathcal{S}}_t + \bar{\boldsymbol{\mathcal{H}}}\boldsymbol{\varepsilon}_{t+1}\,, \qquad \bar{\boldsymbol{\mathcal{T}}} = \begin{bmatrix} \boldsymbol{\mathcal{P}} & \mathbf{0} \\ \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}} & \mathbf{0} \end{bmatrix}, \qquad \bar{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} \boldsymbol{B}_{x\varepsilon}\,, \tag{19}$$

where $\bar{\boldsymbol{\mathcal{T}}}$ is stable because $\boldsymbol{\mathcal{P}}$ is, and the endogenous forecast errors are given by $\boldsymbol{\eta}_t = \boldsymbol{\mathcal{G}}_{yx}\,\boldsymbol{B}_{x\varepsilon}\,\boldsymbol{\varepsilon}_t$.

To construct the coefficient matrices $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ we can follow Klein (2000) and perform a complex generalized Schur decomposition of $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$. The Schur decomposition yields a pair of unitary matrices, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{Z}}$, as well as upper triangular matrices $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{T}}$, such that $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{T}}$.[14] The result is the following triangular system in the so-called canonical variables $\boldsymbol{\mathcal{X}}^*_{t+1}$ and $\boldsymbol{\mathcal{Y}}^*_{t+1}$:

$$\underbrace{\begin{bmatrix} \boldsymbol{\mathcal{U}}_{11} & \boldsymbol{\mathcal{U}}_{12} \\ \mathbf{0} & \boldsymbol{\mathcal{U}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{U}}} E_t \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_{t+1} \\ \boldsymbol{\mathcal{Y}}^*_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{T}}_{11} & \boldsymbol{\mathcal{T}}_{12} \\ \mathbf{0} & \boldsymbol{\mathcal{T}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{T}}} \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_{t} \\ \boldsymbol{\mathcal{Y}}^*_{t} \end{bmatrix}, \tag{A-44}$$

where

$$\begin{bmatrix} \boldsymbol{\mathcal{X}}_{t} \\ \boldsymbol{\mathcal{Y}}_{t} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{Z}}_{11} & \boldsymbol{\mathcal{Z}}_{12} \\ \boldsymbol{\mathcal{Z}}_{21} & \boldsymbol{\mathcal{Z}}_{22} \end{bmatrix}}_{\boldsymbol{\mathcal{Z}}} \begin{bmatrix} \boldsymbol{\mathcal{X}}^*_{t} \\ \boldsymbol{\mathcal{Y}}^*_{t} \end{bmatrix} \tag{A-45}$$

Without loss of generality, the triangular system can always be ordered such that $\boldsymbol{\mathcal{U}}_{11}^{-1}\boldsymbol{\mathcal{T}}_{11}$ is a stable matrix, and $\boldsymbol{\mathcal{U}}_{22}^{-1}\boldsymbol{\mathcal{T}}_{22}$ is an unstable matrix, which designates $\boldsymbol{\mathcal{X}}^*_t$ and $\boldsymbol{\mathcal{Y}}^*_t$ to be stable and unstable canonical variables.

Crucially, our Assumption 1 of a unique stable solution in the full-information case means that the number of stable canonical variables in the vector $\boldsymbol{\mathcal{X}}^*_t$ is identical to the number of backward-

[13]The imperfect information setup could include measurement errors as part of the vector of backward-looking variables, $\boldsymbol{X}_t$. They would affect endogenous variables of the system only via $\boldsymbol{H}$, which does not play a role in the full-information solution. Conceptually, there is no harm including measurement errors in $\boldsymbol{X}_t$, and the corresponding columns of $\boldsymbol{\mathcal{G}}_{yx}$ are equal to zero in this case.

[14]In general, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{Z}}$, $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{T}}$ are complex.

looking variables in $\boldsymbol{\mathcal{X}}_t$ (and the length of the unstable canonical variables' vector $\boldsymbol{\mathcal{Y}}_t^*$ is identical to the length of $\boldsymbol{\mathcal{Y}}_t$). A stable solution of the system then requires the familiar prescription to solve the unstable canonical variables forward and the stable canonical variables backward. The result is the following unique solution:

$$\boldsymbol{\mathcal{Y}}_t^* = \mathbf{0} \tag{A-46}$$

$$\Longrightarrow \qquad \boldsymbol{\mathcal{X}}_t = \boldsymbol{\mathcal{Z}}_{11}\boldsymbol{\mathcal{X}}_t^*$$

$$\Longrightarrow \qquad E_t\boldsymbol{\mathcal{X}}_{t+1} = \underbrace{\boldsymbol{\mathcal{Z}}_{11}\boldsymbol{\mathcal{U}}_{11}^{-1}\boldsymbol{\mathcal{T}}_{11}\boldsymbol{\mathcal{Z}}_{11}^{-1}}_{\boldsymbol{\mathcal{P}}}\boldsymbol{\mathcal{X}}_t \qquad \text{and} \qquad \boldsymbol{\mathcal{Y}}_t = \underbrace{\boldsymbol{\mathcal{Z}}_{21}\boldsymbol{\mathcal{Z}}_{11}^{-1}}_{\boldsymbol{\mathcal{G}}}\boldsymbol{\mathcal{X}}_t \tag{A-47}$$

The defining feature of the unique full-information equilibrium is that non-fundamental disturbances, such as sunspot or belief shocks, do not affect equilibrium outcomes, as discussed, for example, by Lubik and Schorfheide (2003). Hence, the equilibrium is determinate. This is no longer the case under imperfect information.

B.2 Expectation Formation under Imperfect Information

We proceed by making a sequence of assumptions that allow us to map the general framework under asymmetric information into a standard linear rational expectations setting that can be solved and studied with familiar methods. To ensure that a Kalman filter delivers an exact representation of the true conditional expectations, we are interested in linear equilibria, driven by normally distributed disturbances. Therefore, we make the following assumption:

ASSUMPTION 2 (Jointly normal forecast errors) *The endogenous forecast errors are a linear combination of the N_ε exogenous errors, $\boldsymbol{\varepsilon}_t$, and N_y so-called belief shocks, $\boldsymbol{b}_t$, that are mean zero and uncorrelated with $\boldsymbol{\varepsilon}_t$:*

$$\boldsymbol{\eta}_t = \boldsymbol{\Gamma}_\varepsilon\boldsymbol{\varepsilon}_t + \boldsymbol{\Gamma}_b\boldsymbol{b}_t \tag{A-48}$$

Moreover, exogenous shocks and belief shocks are generated from a joint standard normal distribution,

$$\boldsymbol{w}_t \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{b}_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}\right)$$

The matrix of belief shock loadings $\boldsymbol{\Gamma}_b$ need not have full rank so that linear combinations of endogenous and exogenous forecast errors might be perfectly correlated.

As a corollary of Assumption 2, exogenous and endogenous forecast errors are joint normally distributed as well:

$$\begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\eta}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{I} & \boldsymbol{\Gamma}'_\varepsilon \\ \boldsymbol{\Gamma}_\varepsilon & \boldsymbol{\Omega}_\eta \end{bmatrix} \right), \qquad \text{with} \quad \boldsymbol{\Omega}_\eta = \boldsymbol{\Gamma}_b \boldsymbol{\Gamma}'_b + \boldsymbol{\Gamma}_\varepsilon \boldsymbol{\Gamma}'_\varepsilon .$$

In this specification of the endogenous forecast error we follow Lubik and Schorfheide (2003) and Farmer, Khramov, and Nicolò (2015). In a determinate equilibrium, $\boldsymbol{\Gamma}_b$ is the zero matrix, while under indeterminacy it allows for sunspot shocks, or belief shocks, to affect endogenous forecast errors and expectations.[15] As in the full-information literature, Assumption 2 sees endogenous forecast errors as a linear combination of exogenous fundamental shocks and non-fundamental belief shocks. In addition, the belief shocks are assumed to be normally distributed, which is not entirely innocuous in that it restricts the stochastic nature of the equilibrium outcomes. While this assumption is immaterial for the solution under full information, it allows us to derive an exact Kalman-filter representation of expectation formation under imperfect information. Extension to non-Gaussian shocks are beyond the scope of the paper, but constitute important further research.

We now define and characterize a class of stationary, linear and time-invariant equilibria, where expectations are represented by a Kalman filter and where we treat the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ as given. In a subsequent step, we turn to solution methods that determine values for $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with these equilibria.

DEFINITION 1 (Stationary, linear, time-invariant equilibrium) *In a stationary, linear, and time-invariant equilibrium, backward- and forward-looking variables, $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, and the instrument, $\boldsymbol{i}_t$, are stationary and their equilibrium dynamics satisfy the expectational difference system described by* (12) *and* (13)*.*[16] *All expectations are rational, and the imperfectly informed agent's information set is described by* (15)*. In addition, Assumption 2 holds, which means that the forecast errors of the forward-looking variables are a linear combination of fundamental shocks and belief shocks, with time-invariant loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, as in* (A-48) *and all shocks are joint normally distributed.*

We now show that in such an equilibrium the policymaker's conditional expectations can be represented by a Kalman filter. The policymaker observes the measurement equation (15), and the

[15] In a linear equilibrium, where (A-48) holds, the belief shocks affect outcomes only via the product $\boldsymbol{\Gamma}_b\, \boldsymbol{b}_t$. Thus, without loss of generality, $\boldsymbol{b}_t$ can be normalized to have a variance-covariance matrix equal to the identity matrix.

[16] Linear equilibria considered in this paper are driven by normally distributed shocks, leading to normally distributed outcomes, such that covariance stationarity also implies strict stationarity. Hence, we will not distinguish between both concepts and merely refer to stationarity.

state equation of the filtering problem is given by:

$$\boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}}\boldsymbol{S}_{t+1|t} = \boldsymbol{A}\boldsymbol{S}_t + \hat{\boldsymbol{A}}\boldsymbol{S}_{t|t} + \boldsymbol{A}_i\,\boldsymbol{i}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \qquad \text{with} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{\Gamma}_{\varepsilon} & \boldsymbol{\Gamma}_b \end{bmatrix}, \tag{A-49}$$

which combines the expectational difference equation (12) with the implications of Assumption 2 for the endogenous forecast errors. The appearance of projections $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$ in (A-49) lends this state equation a slightly non-standard format. However, when expressed in terms of innovations, the filtering problem can be cast in the canonical "ABCD" form, studied, among others, by Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007):[17]

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{B}\boldsymbol{w}_{t+1}\,, \tag{A-50}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{C}\left(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}\right) + \boldsymbol{D}\boldsymbol{w}_{t+1}\,, \tag{A-51}$$

$$\text{with} \quad \boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}\,, \quad \boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}\,, \quad \text{since} \quad \tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{H}\tilde{\boldsymbol{S}}_{t+1}\,. \tag{A-52}$$

To ensure a well-behaved filtering problem, we impose the following assumption on the shocks to the policymaker's measurement vector $\boldsymbol{D}\boldsymbol{w}_t = \boldsymbol{Z}_t - E_{t-1}\boldsymbol{Z}_t$.

ASSUMPTION 3 (Non-degenerate shocks to the signal equation) *The variance-covariance matrix of the shocks to the measurement equation has full rank: $|\boldsymbol{D}\boldsymbol{D}'| \neq 0$.*

A necessary condition for Assumption 3 to hold is that the signal vector does not have more elements than the sum of endogenous and exogenous forecast errors: $N_z \leq N_\varepsilon + N_y \leq N_x + N_y$.

Existence of a steady-state Kalman filter requires certain conditions on $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ to hold, known in the literature as "observability" and "unit-circle controllability," formally defined in the appendix. As shown in (A-49), $\boldsymbol{B}$ depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$, which are endogenous in our framework. To ensure the existence of a steady-state filter, the following assumption restricts the set of potential equilibria that we consider. But, as discussed further below, these conditions impose only weak restrictions on the shock loadings in $\boldsymbol{B}$.

ASSUMPTION 4 (Sufficient condition for existence of a steady-state Kalman filter) *The equilibrium shock loadings $\boldsymbol{\Gamma}_\varepsilon$, $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$ are such that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are detectable and unit-circle controllable as stated in Definition 5 of Appendix B.4.*

A central component of our framework is that the less informed agent observes data imperfectly, but employs optimal signal extraction, which is provided by the Kalman filter in this linear Gaussian

[17]The innovations form is obtained by projecting both sides of (A-49) onto $\boldsymbol{Z}^t$ and subtracting these projections from (A-49). Note that the control $\boldsymbol{i}_t$ is always in the policymaker's information set. The innovations form given by (A-50) and (A-51) is identical to the innovations form of a state-space system with $\boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}$ in place of (A-49) while maintaining (15) as measurement equation, as noted also by Baxter, Graham, and Wright (2011).

environment. The optimal projections $\boldsymbol{S}_{t|t}$ can be decomposed into the sum of previous period's forecasts $\boldsymbol{S}_{t|t-1}$ and an update that is linear in the innovations in measurement vector. When a steady-state filter exists, a constant Kalman gain relates the projected innovations in the state vector, $\tilde{\boldsymbol{S}}_{t|t}$, to innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$:

$$\boldsymbol{S}_{t|t} = \boldsymbol{S}_{t|t-1} + \tilde{\boldsymbol{S}}_{t|t} \quad \text{with} \quad \tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t\,, \quad \text{and} \quad \boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1}. \tag{A-53}$$

The Kalman gain matrix, $\boldsymbol{K}$, is given by the solution of a standard Riccati equation involving $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{H}$. We detail the derivation in the appendix.

We focus on the case when a steady-state Kalman filter exists, which allows us to represent the policymaker's conditional expectations as a recursive linear system with time-invariant coefficients. We formally state these results in the following two propositions.

PROPOSITION 2 (Existence of steady-state Kalman filter) *When Assumptions 3 and 4 hold in a linear, stationary, and time-invariant equilibrium, a steady-state Kalman filter exists that describes the projection of innovations in the state vector,* $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ *with a constant Kalman gain* $\boldsymbol{K}$ *as in* (A-53)*. The variance-covariance matrix of projection residuals is constant,* $\mathrm{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right) = \mathrm{Var}\left(\boldsymbol{S}_t^*\right) = \Sigma^*$*. Existence of a steady-state Kalman filter ensures that innovations* $\tilde{\boldsymbol{S}}_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t-1}$ *and residuals* $\boldsymbol{S}_t^* = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ *are stationary as are innovations to the measurement equation,* $\tilde{\boldsymbol{Z}}_t$*.*

Proof. See Theorem 2 of Appendix B.4. The stationarity of $\tilde{\boldsymbol{Z}}_t = \boldsymbol{H}\tilde{\boldsymbol{S}}_t$ follows then from the stationarity of $\tilde{\boldsymbol{S}}_t$. ■

Given the existence of a Kalman filter, equation (A-53) describes the optimal update from forecasts, $\boldsymbol{S}_{t|t-1}$, to current projections, $\boldsymbol{S}_{t|t}$. What remains to be characterized is the transition equation from projections $\boldsymbol{S}_{t|t}$ to forecasts, $\boldsymbol{S}_{t+1|t}$. This relationship is restricted by the linear difference equations (12) and the rule (13) for the policy instrument $\boldsymbol{i}_t$. Since the instrument can depend on its own lagged value via $\Phi_i \neq \boldsymbol{0}$ in (13), we construct the transition from $\boldsymbol{S}_{t|t}$ to $\boldsymbol{S}_{t+1|t}$ based on the vector $\boldsymbol{\mathcal{S}}_t$, which includes $\boldsymbol{S}_t$ and $\boldsymbol{i}_{t-1}$, as defined in (16).

We derive the transition equation from projections to forecasts by noting that the structure of the dynamic system conditioned down onto the policymaker's information takes a familiar form. That is, conditioning down (12) and (13) onto $\boldsymbol{Z}^t$ yields a system of expectational linear difference equations in $\boldsymbol{\mathcal{S}}_t$ that is akin to the full-information system shown in (17), except for the use of policymaker projections in lieu of full-information expectations:

$$\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{S}}_{t|t} \tag{A-54}$$

with $\boldsymbol{\mathcal{J}}$, and $\boldsymbol{\mathcal{A}}$ as defined in (17) above. In a stationary equilibrium consistent with Definition 1, $\boldsymbol{\mathcal{S}}_t$ is stationary, and so is its projection $\boldsymbol{\mathcal{S}}_{t|t}$. When the equilibrium is linear and time-invariant, the projections $\boldsymbol{\mathcal{S}}_{t|t}$ must follow

$$\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{S}}_{t|t}\ , \qquad \text{with} \quad (\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{T}} - \boldsymbol{\mathcal{A}})\,\boldsymbol{\mathcal{S}}_{t|t} = 0 \qquad \text{for some stable matrix } \boldsymbol{\mathcal{T}}. \tag{A-55}$$

We can already state another key result in this section. In a linear equilibrium with normally distributed shocks and with a given linear transformation from projections $\boldsymbol{\mathcal{S}}_{t|t}$ into forecasts $\boldsymbol{\mathcal{S}}_{t+1|t}$, the Kalman filter represents conditional expectations of $\boldsymbol{\mathcal{S}}_t$ (and thus also $\boldsymbol{S}_t$). We establish below that in a stationary and linear imperfect information equilibrium that conforms to Definition 1 $\boldsymbol{\mathcal{T}}$ must be identical to its full-information counterpart $\bar{\boldsymbol{\mathcal{T}}}$ as described in equation (19).

PROPOSITION 3 (Kalman filter represents conditional expectations)
In a linear, time-invariant stationary equilibrium, when the conditions for Proposition 2 hold, and for a given stable transition matrix $\boldsymbol{\mathcal{T}}$ between policymaker projections and forecasts as in (A-55)*, the steady-state Kalman filter represents conditional expectations $\boldsymbol{\mathcal{S}}_{t|t} = E\left(\boldsymbol{\mathcal{S}}_t|\boldsymbol{Z}^t\right)$. For a given sequence of innovations in the measurement vector, $\tilde{\boldsymbol{Z}}_t$, the Kalman filter implies a stationary evolution of projections:*

$$\boldsymbol{\mathcal{S}}_{t+1|t+1} = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t} + \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}\ ,$$
$$\textit{with} \quad \boldsymbol{\mathcal{K}} = \operatorname{Cov}\left(\tilde{\boldsymbol{\mathcal{S}}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\operatorname{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \begin{bmatrix} 0' & \boldsymbol{K}_x' & \boldsymbol{K}_y' & \boldsymbol{K}_i' \end{bmatrix}'. \tag{A-56}$$

Proof. In a linear, time-invariant stationary equilibrium, shocks are jointly normal and propagate linearly so that the sequences of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{Z}_t$ are joint normally distributed, and conditional expectations are identical to mean-squared-error optimal linear projections. By the law of iterated projections, we can decompose $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^{t+1}\right) = E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) + E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right)$ and, based on (A-55), we have $E\left(\boldsymbol{\mathcal{S}}_{t+1}|\boldsymbol{Z}^t\right) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{S}}_{t|t}$. From Proposition 2 follows $E\left(\tilde{\boldsymbol{\mathcal{S}}}_{t+1}|\tilde{\boldsymbol{Z}}_{t+1}\right) = \boldsymbol{\mathcal{K}}\tilde{\boldsymbol{Z}}_{t+1}$. $\boldsymbol{K}_x$ and $\boldsymbol{K}_y$ are appropriate partitions of $\boldsymbol{K}$ as defined in (A-53) and $\boldsymbol{K}_i = \boldsymbol{\mathcal{G}}_{ix}\boldsymbol{K}_x$. The upper block of $\boldsymbol{\mathcal{K}}$, corresponding to the gain coefficients for the lagged policy instrument, is zero since $\boldsymbol{i}_{t-1} = \boldsymbol{i}_{t-1|t-1}$ $\Rightarrow \tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$. Projections are stationary since $\boldsymbol{\mathcal{T}}$ is stable. ■

So far, we have characterized $\boldsymbol{\mathcal{T}}$ merely as a stable matrix. We show now that in a stationary and linear imperfect information equilibrium that conforms to Definition 1, the transition matrix $\boldsymbol{\mathcal{T}}$ must be identical to its full-information counterpart $\bar{\boldsymbol{\mathcal{T}}}$ as described in equation (19). Specifically, we begin by defining a set of equilibrium conditions labelled a "projection condition" that imply $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$. We then proceed to establish that the projection condition needs to hold for any equilibrium that conforms to Definition 1.

DEFINITION 2 (Projection Condition) *The projection condition restricts the mapping between projected backward- and forward-looking variables to be identical to the full-information case:*

$$\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}\,, \qquad\qquad \text{and} \quad \boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t}\,. \tag{A-57}$$

where $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{P}}$ are the unique solution coefficients in the corresponding full-information case.

The projection condition is an equilibrium condition that imposes a linear mapping between projections of backward- and forward-looking variables.[18] Moreover, the projection condition is required to hold in any stationary equilibrium that is also linear and time-invariant.

PROPOSITION 4 (Projection Condition holds when full-information case is determinate) *If Assumption 1 (uniqueness of full-information equilibrium) holds, the projection condition must hold in any stationary linear and time-invariant imperfect information equilibrium that conforms to Definition 1. It follows that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$.*

Proof. As noted before, conditioning down (12) and (13) onto $\boldsymbol{Z}^t$ yields a system of expectational linear difference equations that is akin to the full-information system shown in (17), except for the use of policymaker projections in lieu of full-information expectations: $\boldsymbol{\mathcal{J}} E_t \boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}} \boldsymbol{\mathcal{S}}_{t|t}$. Since $\boldsymbol{\mathcal{J}}$ and $\boldsymbol{\mathcal{A}}$ are identical to the full-information case, we obtain the same complex generalized Schur decomposition as in the full-information case of equations (A-44) and (A-45), with the same partitioning into stable and unstable canonical variables (except for expressing these in terms of projections, $\boldsymbol{\mathcal{X}}^*_{t|t}$ and $\boldsymbol{\mathcal{Y}}^*_{t|t}$). As argued above, the transition equation for projections must be stable to ensure a stationary equilibrium. In terms of the canonical variables, this requires $\boldsymbol{\mathcal{Y}}^*_{t|t} = \mathbf{0}$ and following steps analogous to the full-information case as in (A-46) – (A-47) we obtain the projection condition. Given the definition of $\bar{\boldsymbol{\mathcal{T}}}$ in (19), it follows that $\boldsymbol{\mathcal{T}} = \bar{\boldsymbol{\mathcal{T}}}$ when the projection condition holds. ■

Moreover, the projection condition imposes a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{Y}}_t$.[19] As a second-moment restriction, the projection condition only restricts co-movements of the innovations *on average*, but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$. The projection condition (20) implies the following restriction between the Kalman gains of forward-

[18] In a setup similar to ours, Svensson and Woodford (2004) also embed a set of equations akin to the projection condition, based on an appeal to certainty equivalence. In contrast to our paper, they restrict the number of state variables in a way that disregards a role for belief shocks, and assume equilibrium uniqueness. Applications that build on Svensson and Woodford (2004) are, for example, Dotsey and Hornstein (2003), Aoki (2006), Nimark (2008), Carboni and Ellison (2011).

[19] In addition to $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, $\boldsymbol{\mathcal{Y}}_t$ and $\boldsymbol{\mathcal{X}}_t$ also contain the current and lagged policy instrument, respectively. However, the projection condition does not impose a direct restriction on innovations in the policy instrument since $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ and thus $\tilde{\boldsymbol{i}}_{t-1|t} = \boldsymbol{i}_{t-1|t} - \boldsymbol{i}_{t-1|t-1} = 0$.

and backward-looking variables:

$$\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{G}}_{yi}\ \boldsymbol{i}_{t-1|t} \quad \Rightarrow \tilde{\boldsymbol{Y}}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \tilde{\boldsymbol{X}}_{t|t} \quad \Leftrightarrow \quad (\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x)\,\tilde{\boldsymbol{Z}}_t = \boldsymbol{0}\,, \quad \text{(A-58)}$$

where $\boldsymbol{K}_y$ and $\boldsymbol{K}_x$ denote the corresponding partitions of the Kalman gain, $\boldsymbol{K}$, defined in (A-53). Since equation (A-58) must hold for every $\tilde{\boldsymbol{Z}}_t$, the projection condition therefore implies a restriction on the Kalman gains, which we summarize in the following proposition.

PROPOSITION 5 (Projection Condition for Kalman gains) *The projection condition* (20) *holds only if the Kalman gains satisfy* $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x$.

Proof. As noted in (A-58), a necessary condition for the projection to hold is $(\boldsymbol{K}_y - \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x)\,\tilde{\boldsymbol{Z}}_t = 0$ for all realizations of $\tilde{\boldsymbol{Z}}_t$. $\tilde{\boldsymbol{Z}}_t$ has a joint normal distribution and Assumption 3 implies that $\text{Var}\,(\tilde{\boldsymbol{Z}}_t) = \boldsymbol{C}\,\text{Var}\,(\boldsymbol{S}^*_t)\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'$ is strictly positive definite, so that the distribution of $\tilde{\boldsymbol{Z}}_t$ is non-degenerate. Thus, for (A-58) to hold we must have $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x$. Since

$$\boldsymbol{K}_y = \text{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t)\,\text{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}\,, \text{ and } \boldsymbol{K}_x = \text{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)\,\text{Var}\,(\tilde{\boldsymbol{Z}}_t)^{-1}\,,$$

the projection condition equivalently requires $\text{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\ \text{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)$. ■

Intuitively, the Kalman gains are multivariate regression slopes and, as shown in Proposition 5, the projection condition imposes a linear restriction on covariances between $\tilde{\boldsymbol{Y}}_t$ and $\tilde{\boldsymbol{X}}_t$. It is this feature of our framework that moves the solution of the underlying linear rational expectations model out of certainty equivalence and poses an intricate fixed-point problem with a highly non-linear solution: The Kalman gains depend on the second moments of the solution, which in turn depends on the Kalman gains via the projection condition.

We are now in a position to derive a representation of the system of expectational difference equations (12) - (14), where the expectations of the fully informed agent are captured by the concept of endogenous forecast errors known from Definition 2 and the expectation formation of the less-informed agent is represented by the Kalman filter. As summarized in the following theorem, equilibrium dynamics then follow a state vector that tracks both projections and actual values of backward- and forward-looking variables as well as the policy instrument.[20]

THEOREM 1 (Difference System Under Imperfect Information) *Consider the system of difference equations* (12) *and* (13) *with a measurement vector that is linear in* $\boldsymbol{S}_t$ *as defined in* (15).

[20]In fact, the joint vector of $\boldsymbol{\mathcal{S}}_t$ and $\boldsymbol{\mathcal{S}}_{t|t}$ does not need to tracked in its entirety. First, $\boldsymbol{\mathcal{S}}_t$ already includes the policy instrument $\boldsymbol{i}_t$, which lies in the space of central bank projections. The state of the economy can thus be described by $\boldsymbol{\mathcal{S}}_{t|t}$ and $\boldsymbol{S}_t$, which differs from $\boldsymbol{\mathcal{S}}_t$ in omitting $\boldsymbol{i}_t$. Second, the state of the economy is equivalently described by $\boldsymbol{S}^*_t = \boldsymbol{S}_t - \boldsymbol{S}_{t|t}$ and $\boldsymbol{\mathcal{S}}_{t|t}$. Third, when the projection condition (20) is satisfied, we need only track $\boldsymbol{\mathcal{X}}_{t|t}$ rather than $\boldsymbol{\mathcal{S}}'_{t|t} = \begin{bmatrix}\boldsymbol{\mathcal{X}}'_{t|t} & \boldsymbol{\mathcal{Y}}'_{t|t}\end{bmatrix}$, since $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$.

Let Assumptions 1, 2, 3, and 4 hold and consider stationary, linear, time-invariant equilibria that satisfy the projection condition, as stated in Definitions 1 and 2. Equilibrium dynamics are then characterized by the evolution of the following vector system:

$$\overline{\boldsymbol{\mathcal{S}}}_{t+1} \equiv \begin{bmatrix} \boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{K}\boldsymbol{C}) & 0 \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{C} & \boldsymbol{\mathcal{P}} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_t + \begin{bmatrix} (\boldsymbol{I}-\boldsymbol{K}\boldsymbol{H}) \\ \boldsymbol{\mathcal{K}}_x\boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1} \end{bmatrix},$$

where $\boldsymbol{\mathcal{K}}_x' = \begin{bmatrix} 0 & \boldsymbol{K}_x' \end{bmatrix}$, $\boldsymbol{C}$ *as defined in* (A-52)*, and* $\boldsymbol{\mathcal{P}}$ *known from the unique full-information solution.*

Proof. $\boldsymbol{S}_t$ can be decomposed into $\boldsymbol{S}_t = \boldsymbol{S}^*_t + \boldsymbol{S}_{t|t}$. We need to show that $\boldsymbol{S}_{t|t}$ can be constructed from $\overline{\boldsymbol{\mathcal{S}}}_t$, and that the policy instrument $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ can be constructed from the proposed state vector $\overline{\boldsymbol{\mathcal{S}}}_t$. Recalling the definitions of $\boldsymbol{S}_t$, $\boldsymbol{\mathcal{X}}_t$ and $\boldsymbol{\mathcal{Y}}_t$ in (14), and (16), the projection condition then implies that:

$$\boldsymbol{S}_{t|t} = \begin{bmatrix} \boldsymbol{X}_{t|t} \\ \boldsymbol{Y}_{t|t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_S} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \boldsymbol{i}_t = \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix}}_{\boldsymbol{\mathcal{G}}_i} \boldsymbol{\mathcal{X}}_{t|t}, \qquad \text{with} \quad \boldsymbol{\mathcal{X}}_{t|t} = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_{t|t} \end{bmatrix},$$

where block matrices are partitioned along the lines of $\boldsymbol{\mathcal{X}}_{t|t}$ above. The various coefficient matrices "$\boldsymbol{\mathcal{G}}_{..}$" are known from the full-information solution given in (18). The dynamics of $\boldsymbol{S}^*_{t+1}$, as captured by the top rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from the innovation state space (A-50) - (A-51) and the steady-state Kalman filter described in Appendix B.4. The dynamics of $\boldsymbol{\mathcal{X}}_{t+1|t+1}$, as captured by the bottom rows of $\overline{\boldsymbol{\mathcal{S}}}$, follow from (A-56) together with the projection condition (20) and the dynamics of $\tilde{\boldsymbol{Z}}_t$ given in (A-51). ■

Equation (21) forms the basis for solving a rational expectations model with heterogeneous information sets. Based on this representation we can derive key insights into the nature of the equilibria in this setting. The state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ follows a first-order linear difference system given in (21). The stability of the system depends on the eigenvalues of its transition matrix $\overline{\boldsymbol{\mathcal{A}}}$. The transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ depends on the Kalman gain $\boldsymbol{K}$, which depends on the yet to be determined shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_t$. Nevertheless, existence of a steady-state Kalman filter allows us to conclude that $\overline{\boldsymbol{\mathcal{A}}}$, is stable.

COROLLARY 1 (Stable Transition Matrix) *Provided that a steady-state Kalman filter exists, the transition matrix* $\overline{\boldsymbol{\mathcal{A}}}$ *in* (21) *is stable. The eigenvalues of* $\overline{\boldsymbol{\mathcal{A}}}$ *are given by the eigenvalues of* $\boldsymbol{\mathcal{P}}$*, which is stable and known from the full-information solution* (18)*, and* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$*, whose stability is assured by the existence of the steady-state Kalman filter.*

Proof. The stability of $\mathcal{P}$ follows from Assumption 1 and the resulting solution of the full-information case. The stability of $\boldsymbol{A} - \boldsymbol{KC}$ follows from Theorem 2 in Appendix B.4. ■

One key implication of Proposition 1 is that the usual root-counting arguments, as in Blanchard and Kahn (1980) or generalized in Sims (2002), do not pin down the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ of the endogenous forecast errors $\boldsymbol{\eta}_{t+1}$ in (21). This follows from the fact that $\overline{\boldsymbol{\mathcal{A}}}$ is a stable matrix for any choice of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ consistent with the existence of a steady-state Kalman filter. Moreover, the projection condition does typically not place sufficiently many restrictions on $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ to uniquely identify the shock loadings. This observation is the key result of our paper, namely that equilibria are indeterminate.

PROPOSITION 6 (Indeterminacy) *With $\overline{\boldsymbol{\mathcal{A}}}$ stable, the endogenous forecast errors are only restricted by the projection condition given in Definition 2. The shock loadings of the endogenous forecast errors, $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, have $N_y \times (N_\varepsilon + N_y)$ unknown conditions. The projection condition stated in* (A-58) *imposes only $N_y \times N_z$ restrictions. However, a necessary condition for Assumption 3 to hold is $N_z \leq N_\varepsilon + N_y$. As a result, the projection condition cannot uniquely identify the shock loadings $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$.*

To summarize, in this section we derive two key insights for our linear rational expectations framework with heterogeneous information sets. First, we show that under Gaussian shocks a Kalman filter exists, which is the optimal filter in this environment. The filter is available to the less-informed agent to gain information about realizations of variables that are available to the fully informed agent. Second, we show that the Kalman filter represents the conditional expectations of the less informed agent. This insight allows us to construct a solution approach for this framework which deviates from standard methods in that certainty equivalence no longer holds. Specifically, expectation formation of the less informed agents now depends on second moments embedded in the steady-state Kalman filter. Moreover, it is the intersection of the different expectation formation processes under rational expectations that lies at the heart of indeterminacy in this framework.

B.3 Determination of the Endogenous Forecast Errors

In the full-information literature, restrictions for the endogenous forecast errors $\boldsymbol{\eta}_t$ emanate from explosive roots in the dynamic system, as described by Sims (2002) and Lubik and Schorfheide (2003). In our imperfect information case, further restrictions result from the projection condition stated in equation 20. As noted in Corollary 1 above, the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ is always stable in a time-invariant equilibrium with a steady-state Kalman filter. Restrictions on $\boldsymbol{\eta}_t$ can therefore only result from the projection condition. At the same time, the projection condition does generally not provide sufficiently many restrictions to pin down $\boldsymbol{\eta}_t$ uniquely, as discussed in Proposition 6.

Determination of shock loadings $\mathbf{\Gamma}_\varepsilon$, $\mathbf{\Gamma}_b$ for the endogenous forecast errors that are consistent with the projection condition poses an intricate fixed problem between shock loadings and Kalman gains. As noted already by Sargent (1991), the Kalman gains are endogenous equilibrium objects when the observable signals reflect information contained in endogenous variables. In contrast, as discussed in an earlier working paper version Lubik, Matthes, and Mertens (2019), the Kalman filtering problem can be solved independently of the equilibrium dynamics of the system when the signal vector consists only of exogenous variables.[21]

Equation (15) describes the central bank's measurement vector generically as a linear combination of backward- and forward-looking variables, $\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{H}_y\boldsymbol{Y}_t$. To simplify some of the algebra, we limit ourselves to signal vectors that have the same length as the vector of forward-looking variables ($\boldsymbol{Y}_t$) and that have no rank-deficient loading on $\boldsymbol{Y}_t$, so that $\boldsymbol{H}_y$ is square and invertible, and can be normalized to the identity matrix:[22]

$$\boldsymbol{Z}_t = \boldsymbol{H}_x\boldsymbol{X}_t + \boldsymbol{Y}_t \qquad \text{and thus} \quad \boldsymbol{H} = \begin{bmatrix}\boldsymbol{H}_x & \boldsymbol{I}\end{bmatrix}. \tag{A-59}$$

This setup also includes the case where each forward-looking variable is observed with error, as in $\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t$, where $\boldsymbol{\nu}_t$ is an exogenous measurement error to be included among the set of backward-looking variables in $\boldsymbol{X}_t$.

We pursue a numerical the solution for the shock loadings $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ that are consistent with the projection condition. Our procedure combines conventional techniques for solving linear RE models with standard algorithms for solving the Kalman filter's Riccati equation, while ensuring consistency with the projection condition. The algorithm searches for shock loadings $\mathbf{\Gamma}_\varepsilon$ and $\mathbf{\Gamma}_b$ that satisfy the projection condition for a Kalman filter that is consistent with equilibrium outcomes of all variables.

From the measurement equation (A-59), we can deduce that $\boldsymbol{Y}_t^* = -\boldsymbol{H}_x\boldsymbol{X}_t^*$, which allows to simplify the innovation system given by (A-50) and (A-51) as follows: $\tilde{\boldsymbol{X}}_{t+1} = \tilde{\boldsymbol{A}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{B}}\boldsymbol{w}_{t+1}$, and $\tilde{\boldsymbol{Z}}_{t+1} = \tilde{\boldsymbol{C}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{D}}\boldsymbol{w}_{t+1}$ with $\tilde{\boldsymbol{A}} = \boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x$, $\tilde{\boldsymbol{B}} = \begin{bmatrix}\boldsymbol{B}_{x\varepsilon} & 0\end{bmatrix}$, $\tilde{\boldsymbol{C}} = \boldsymbol{H}_x\ (\boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\boldsymbol{H}_x)+$

[21]The case of a purely exogenous signal arises when $\boldsymbol{H}_y = \boldsymbol{0}$ in (15) and $\boldsymbol{X}_t$ exogenous (i.e. without dependence on lagged forward-looking variables). The working paper version of this article contains further discussion and a general analytical characterization of the solution for this case. In Lubik, Matthes, and Mertens (2019), we derive two general results for the exogenous-signal case: First, the projection condition does not restrict the belief shock loadings of the endogenous forecast errors, $\mathbf{\Gamma}_b$, when the signal is exogenous. Second, we derive an analytical expression for the restrictions on the loadings of the endogenous forecast errors on fundamental shocks (including the measurement errors) that result from the projection condition. However, the exogenous-signal case is arguably less realistic since variables relevant to the policymaker are typical endogenous.

[22]Consider the case of a signal $\hat{\boldsymbol{Z}}_t = \hat{\boldsymbol{H}}_x\boldsymbol{X}_t + \hat{\boldsymbol{H}}_y\boldsymbol{Y}_t$, where $\hat{\boldsymbol{H}}_y$ is square and invertible. The information content provided by $\hat{\boldsymbol{Z}}_t$ is equivalent to what is spanned by $\boldsymbol{Z}_t = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{Z}}_t$ with $\boldsymbol{H}_x = \hat{\boldsymbol{H}}_y^{-1}\hat{\boldsymbol{H}}_x$.

$\boldsymbol{A}_{yx} - \boldsymbol{A}_{yy}\,\boldsymbol{H}_x$,[23]

$$\text{and} \quad \tilde{\boldsymbol{D}} = \begin{bmatrix} (\boldsymbol{H}_x \boldsymbol{B}_{x\varepsilon} + \boldsymbol{\Gamma}_\varepsilon) & \boldsymbol{\Gamma}_b \end{bmatrix} . \tag{A-60}$$

For a given $\tilde{\boldsymbol{D}}$, which embodies a guess of $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$, the Kalman-filtering solution to this system generates a Kalman gain $\boldsymbol{K}_x$, which can be used to form projections $\tilde{\boldsymbol{X}}_{t|t} = \boldsymbol{K}_x \tilde{\boldsymbol{Z}}_t$. The algorithm checks whether this guess for $\tilde{\boldsymbol{D}}$ also satisfies the projection condition, which requires $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx} \boldsymbol{K}_x$. Together with the projection condition, the measurement equation (A-59) implies $\boldsymbol{I} = \boldsymbol{H}_x \boldsymbol{K}_x + \boldsymbol{K}_y = (\boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx})\,\boldsymbol{K}_x$. We employ a numerical solver that searches for a $\tilde{\boldsymbol{D}}$ that generates a Kalman gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, where $\boldsymbol{L} = \boldsymbol{H}_x + \boldsymbol{\mathcal{G}}_{yx}$. Given a solution for $\tilde{\boldsymbol{D}}$ that satisfies the projection condition $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, we can then back out $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_b$ based on (A-60).

B.4 The Steady State Kalman Filter

This section describes details of the steady-state Kalman filter for the innovations state space (A-50) and (A-51) when Assumption 3 holds. Existence of a steady-state Kalman filter relies on finding an ergodic distribution for $\boldsymbol{S}_t^*$ (and thus $\tilde{\boldsymbol{S}}_t$) with constant second moments $\boldsymbol{\Sigma} \equiv \mathrm{Var}(\boldsymbol{S}_t^*)$. When a steady-state filter exists, a constant Kalman gain, $\boldsymbol{K}$ relates projected innovations of $\tilde{\boldsymbol{S}}_t$ to innovations in the signal, $\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t$ with:[24]

$$\boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = (\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}')\,(\boldsymbol{C}\Sigma\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}')^{-1} .$$

The dynamics of $\boldsymbol{S}_t^*$ are then characterized by

$$\boldsymbol{S}_{t+1}^* = (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\;\boldsymbol{S}_t^* + (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D})\,\boldsymbol{w}_{t+1} \tag{A-61}$$

Existence of a steady-state filter depends on finding a symmetric, positive (semi) definite solution $\boldsymbol{\Sigma}$ to the following Riccati equation:

$$\begin{aligned} \Sigma &= (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\Sigma(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})' + \boldsymbol{B}\boldsymbol{B}' \\ &= \boldsymbol{A}\Sigma\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - (\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}')\,(\boldsymbol{C}\Sigma\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}')^{-1}\,(\boldsymbol{A}\Sigma\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}')' . \end{aligned} \tag{A-62}$$

Intuitively, the Kalman filter seeks to construct mean-squared error optimal projections $\boldsymbol{S}_{t|t}$ that minimize Σ. A necessary condition for the existence of a solution to this minimization problem is the ability to find at least some gain $\hat{\boldsymbol{K}}$ for which $\boldsymbol{A} - \hat{\boldsymbol{K}}\boldsymbol{C}$ is stable; otherwise, $\boldsymbol{S}^*$ would

[23] $\boldsymbol{A}_{xx}$, $\boldsymbol{A}_{xy}$, etc. denote suitable sub-matrices of $\boldsymbol{A}$.

[24] See also (A-53) in the paper.

have unstable dynamics as can be seen from (A-61). Thus, existence of the second moment for the residuals, $\text{Var}\left(\boldsymbol{S}_t^*\right) = \boldsymbol{\Sigma} \geq \mathbf{0}$, is synonymous with a stable transition matrix $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$.

Formal conditions for the existence of a time-invariant Kalman filter have been stated, among others, by Anderson and Moore (1979), Anderson, McGrattan, Hansen, and Sargent (1996), Kailath, Sayed, and Hassibi (2000), and Hansen and Sargent (2007). Necessary and sufficient conditions for the existence of a unique and stabilizing solution that is also positive semi-definite depend on the "detectability" and "unit-circle controllability" of certain matrices in our state space. We restate those concepts next.

DEFINITION 3 (Detectability) *A pair of matrices $(\boldsymbol{A}, \boldsymbol{C})$ is detectable when no right eigenvector of $\boldsymbol{A}$ that is associated with an unstable eigenvalue is orthogonal to the row space of $\boldsymbol{C}$. That is, there is no non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda$ and $|\lambda| \geq 1$ with $\boldsymbol{C}\boldsymbol{v} = \mathbf{0}$.*

Detectability alone is already sufficient for the existence of *some* solution to the Riccati equation such that $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is stable; see Kailath, Sayed, and Hassibi (2000, Table E.1). Evidently, detectability is assured when $\boldsymbol{A}$ is a stable matrix, regardless of $\boldsymbol{C}$. To gain further intuition for the role of detectability, consider transforming $\boldsymbol{S}_t$ into "canonical variables" by premultiplying $\boldsymbol{S}_t$ with the matrix of eigenvectors of $\boldsymbol{A}$ — this transformation into canonical variables is at the heart of procedures for solving rational expectations models known from Blanchard and Kahn (1980), King and Watson (1998), Klein (2000), Sims (2002). Detectability then requires the signal equation (A-51) to provide some signal (i.e. to have non-zero loadings) for any unstable canonical variables.[25]

To establish existence of a solution to the Riccati equation that is unique and positive semi-definite, we follow Kailath, Sayed, and Hassibi (2000) and require unit-circle controllability, defined as follows.

DEFINITION 4 (Unit-circle controllability) *The pair $(\boldsymbol{A}, \boldsymbol{B})$ is unit-circle controllable when no left-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue on the unit circle is orthogonal to the column space of $\boldsymbol{B}$. That is, there is no non-zero row vector $\boldsymbol{v}$ such that $\boldsymbol{v}\boldsymbol{A} = \boldsymbol{v}\lambda$ with $|\lambda| = 1$ and $\boldsymbol{v}\boldsymbol{B} = \mathbf{0}$.*

In our state space, with $\boldsymbol{B}\boldsymbol{D}' \neq \mathbf{0}$, shocks to state and measurement equation are generally

[25] Specifically, let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ with $\boldsymbol{\Lambda}$ diagonal be the eigenvalue-eigenvector factorization of $\boldsymbol{A}$ so that the columns of $\boldsymbol{V}$ correspond to the right eigenvectors of $\boldsymbol{A}$. Define canonical variables $\boldsymbol{S}_t^C \equiv \boldsymbol{V}^{-1}\boldsymbol{S}_t$. The signal equation can then be stated as $\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{V}\boldsymbol{S}_t^C$ and detectability requires the signal equation to have non-zero loadings on at least every canonical variable associated with an unstable eigenvalue in $\boldsymbol{\Lambda}$.

correlated. Unit-circle controllability is thus applied to the following transformations of $\boldsymbol{A}$, $\boldsymbol{B}$:[26]

$$\boldsymbol{A}^C \equiv \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{C} \qquad\qquad \boldsymbol{B}^C \equiv \boldsymbol{B}\left(\boldsymbol{I} - \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D}\right)$$

Based on these definitions, the following theorem restates results from Kailath, Sayed, and Hassibi (2000) in our notation:

THEOREM 2 (Stabilizing Solution to Riccati Equation) *Provided Assumption 3 holds, a stabilizing and positive semi-definite solution to the Riccati equation* (A-62) *exists when* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable and* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable. The steady-state Kalman gain is such that* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ *is a stable matrix; moreover, the stabilizing solution is unique.*[27]

Proof. See Theorem E.5.1 of Kailath, Sayed, and Hassibi (2000); related results are also presented in Anderson, et al. (1996), or Chapter 4 of Anderson and Moore (1979). ∎

In our context, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the conditions for detectability and unit-circle controllability can also be restated as follows.

PROPOSITION 7 (Detectability of $(\boldsymbol{A}, \boldsymbol{H})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$*, detectability of* $(\boldsymbol{A}, \boldsymbol{C})$ *is equivalent to detectability of* $(\boldsymbol{A}, \boldsymbol{H})$

Proof. When $(\boldsymbol{A}, \boldsymbol{C})$ are detectable, we have $\boldsymbol{C}\boldsymbol{v} \neq \boldsymbol{0}$ for any right-eigenvector of $\boldsymbol{A}$ associated with an eigenvalue λ on or outside the unit circle, $|\lambda| \geq 1$. With $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ we then also have $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{A}\boldsymbol{v} = \boldsymbol{H}\boldsymbol{v}\lambda \neq 0 \Leftrightarrow \boldsymbol{H}\boldsymbol{v} \neq 0$ ∎

Furthermore, with $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ and $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$, the above expressions for $\boldsymbol{A}^C$ and $\boldsymbol{B}^C$ can be transformed as follows:

$$\boldsymbol{A}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{B} \quad \text{with} \quad \boldsymbol{P}^C \equiv \boldsymbol{B}\boldsymbol{H}' \left(\boldsymbol{H}\boldsymbol{B}\boldsymbol{B}'\boldsymbol{H}'\right)^{-1} \boldsymbol{H}. \tag{A-63}$$

$\boldsymbol{P}^C$ is a non-symmetric, idempotent projection matrix with $\boldsymbol{H}\boldsymbol{P}^C = \boldsymbol{H}$.[28]

PROPOSITION 8 (Unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$) *With* $\boldsymbol{C} = \boldsymbol{H}\boldsymbol{A}$ *and* $\boldsymbol{D} = \boldsymbol{H}\boldsymbol{B}$*, unit-circle controllability of* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is equivalent to unit-circle controllability of* $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ *with* $\boldsymbol{P}^C$ *defined in* (A-63)*.*

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[26]Notice that $\boldsymbol{B}^C = \boldsymbol{B}\boldsymbol{\mathcal{M}}^D$ where $\boldsymbol{\mathcal{M}}^D = \boldsymbol{I} - \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D}$ is a projection matrix, which is symmetric and idempotent, $\boldsymbol{\mathcal{M}}^D = \boldsymbol{\mathcal{M}}^D\boldsymbol{\mathcal{M}}^D$, and orthogonal to the row space of $\boldsymbol{D}$.To appreciate the role of $\mathcal{M}^D$, consider the following thought experiment: $\boldsymbol{\mathcal{M}}^D$ construct the residual in projecting the shocks of the system *off* the shocks in the signal equation, $\boldsymbol{w}_t - E(\boldsymbol{w}_t|\boldsymbol{D}\boldsymbol{w}_t) = \boldsymbol{\mathcal{M}}^D\boldsymbol{w}_t$.

[27]There may be other, non-stabilizing positive semi-definite solutions.

[28]An idempotent matrix is equal to its own square, that is $\boldsymbol{P}^C = \boldsymbol{P}^C\boldsymbol{P}^C$, and the eigenvalues of an idempotent matrix are either zero or one and we have $|\boldsymbol{P}^C| = \boldsymbol{0}$.

Proof. Suppose $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ are unit-circle controllable. Let $\tilde{\boldsymbol{v}} \equiv \boldsymbol{v}(\boldsymbol{I} - \boldsymbol{P}^C)$ and note that left-eigenvectors of $\boldsymbol{A}^C$ associated with eigenvalues on the unit circle cannot be orthogonal to $\boldsymbol{P}^C$ (otherwise we would have $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{0}$). Accordingly, $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ with $|\lambda| = 1$, $\boldsymbol{v}\boldsymbol{B}^C \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$ is equivalent to $\tilde{\boldsymbol{v}}\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C) = \tilde{\boldsymbol{v}}\lambda$ with $|\lambda| = 1$, $\tilde{\boldsymbol{v}} \neq \boldsymbol{0}$ $\tilde{\boldsymbol{v}}\boldsymbol{B} \neq \boldsymbol{0}$. The converse reasoning applies as well. ■

As discussed in the main text, an upshot of Proposition 8 is that a sufficient condition for unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is for $\boldsymbol{B}$ to have full rank. Finally, for convenience, we define the the joint concept of detectability and unit-circle controllability for the triplet $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H})$.

DEFINITION 5 (Joint detectability and unit-circle controllability) *The triplet $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$ is detectable and unit-circle controllable when $(\boldsymbol{A}, \boldsymbol{H})$ is detectable and $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$ is unit-circle controllable, where $\boldsymbol{P}^C$ is defined in* (A-63).

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