Appendix to Two-sided Learning and Short-Run Dynamics in a New Keynesian Model*

Christian Matthes† Francesca Rondina‡

Matrices in the PLMs for private agents and the central bank

The true model of the economy is a standard New Keynesian framework, as developed by Gali (2008); the specification that we use is similar to Dennis and Ravenna (2008). We assume perfect indexation of prices that cannot be reset to past inflation, as in Christiano et al. (2001).[1]

Private agents' behavior in this economy can be described by the following equations:

$$y_t = E_t^{PA}\left(y_{t+1}\right) - \frac{1}{\sigma}\left(i_t - E_t^{PA}\left(\pi_{t+1}\right) - r_t^n\right) \tag{1}$$

$$\pi_t = \frac{1}{(1+\beta)}\pi_{t-1} + \frac{\beta}{(1+\beta)}E_t^{PA}\left(\pi_{t+1}\right) - \frac{\kappa}{(1+\beta)}y_t + w_t \tag{2}$$

$$r_t^n = \overline{r} + u_t \tag{3}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u \tag{4}$$

$$w_t = \rho_w w_{t-1} + \varepsilon_t^w \tag{5}$$

where all the variables are as described in the paper. In addition the central bank controls the nominal interest rate i_t through the policy instrument x_t according to the equation:

$$i_t = x_t + v_t \tag{6}$$

where v_t is a monetary policy shocks, which is assumed to follow the AR(1) process:

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v \tag{7}$$

*The views expressed in this appendix are those of the authors and don't necessarily reflect the position of the Federal Reserve Bank of Richmond or the Federal Reserve System.

†Federal Reserve Bank of Richmond, email: christian.matthes@rich.frb.org

‡University of Ottawa, email: frondina@uottawa.ca

[1]This assumption ensures that the pricing equations are unaffected by the presence of positive trend inflation, so that the steady state output level is independent of the steady state inflation level. See Ascari (2004) for a discussion.

Private agents estimate:

$$i_t = z_{t-1}^{R\prime}\psi_t + \omega_t^{PA} \tag{8}$$

while the central bank estimates:

$$y_t = z_{t-1}^{R\prime} c_{yt} + \omega_{yt}^{CB} \tag{9}$$

$$\pi_t = z_{t-1}^{R\prime} c_{\pi t} + \omega_{\pi t}^{CB} \tag{10}$$

where in both cases:

$$z_t^R = \begin{bmatrix} y_t & \pi_t & i_t & 1 \end{bmatrix}' \tag{11}$$

The central bank, chooses the policy rule for x_t by minimizing the expected discounted quadratic loss function:

$$E_{t-1}^{CB} \sum_{j=0}^{\infty} \beta^j [(\pi_{t+j})^2 + \lambda_y (y_{t+j})^2 + \lambda_i (i_{t+j} - i_{t+j-1})^2] \tag{12}$$

given (9) and (10), and the estimated values of c_{yt} and $c_{\pi t}$. Since the central bank's optimization problem is repeated in every period given updated values of c_{yt} and $c_{\pi t}$, the optimal policy vector will be dependent on the current period estimates of these parameters and it will be changing over time: $x_t = -F_t z_{t-1}^R$. The expression for the nominal interest rate becomes:

$$i_t = f_{0t} + f_{\pi t}\pi_{t-1} + f_{yt} y_{t-1} + f_{it} i_{t-1} + v_t \tag{13}$$

The matrices of the PLM for the central bank can easily be obtained using (9), (10), and the policy rule (13).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_t \\ \pi_t \\ i_t \\ v_t \\ 1 \end{pmatrix} = \begin{pmatrix} c_{1yt} & c_{2yt} & c_{3yt} & 0 & c_{4yt} \\ c_{1\pi t} & c_{2\pi t} & c_{3\pi t} & 0 & c_{4\pi t} \\ -f_{\pi t} & -f_{yt} & -f_{it} & 0 & -f_{0t} \\ 0 & 0 & 0 & \rho_v & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \\ v_{t-1} \\ 1 \end{pmatrix} + \\ + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_{yt}^{CB} \\ \omega_{\pi t}^{CB} \\ \varepsilon_t^v \end{pmatrix}$$

or:

$$A^{CB} z_t^{CB} = \left(C_t^{CB} - B^{CB} F_t\right) z_{t-1}^{CB} + D^{CB} \varepsilon_t^{CB}$$

This expression gives the PLM for the Central Bank:

$$z_t^{CB} = \Phi_{1,t} z_{t-1}^{CB} + \Phi_2 \varepsilon_t^{CB} \tag{14}$$

where $\Phi_{1,t} = \left(A^{CB}\right)^{-1}\left(C_t^{CB} - B^{CB} F_t Q^{CB}\right)$ and $\Phi_2 = \left(A^{CB}\right)^{-1} D^{CB}$.

The PLM for private agents is given by the equations of the model $(1)-(4)$ together with the perceived interest rate rule (8). These equations can be rewritten in matrix form as:

$$\begin{pmatrix} 1 & 0 & \frac{1}{\sigma} & -\frac{1}{\sigma} & 0 & 0 & 0 & -\frac{1}{\sigma}\overline{r} \\ \frac{\kappa}{(1+\beta)} & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_t \\ \pi_t \\ i_t \\ u_t \\ w_t \\ E_t^{PA}(y_{t+1}) \\ E_t^{PA}(\pi_{t+1}) \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & \frac{1}{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta}{(1+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} E_t^{PA} \left[\begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \\ u_{t+1} \\ w_{t+1} \\ E_{t+1}^{PA}(y_{t+2}) \\ E_{t+1}^{PA}(\pi_{t+2}) \\ 1 \end{pmatrix} \right] +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_{yt} & \psi_{\pi t} & \psi_{it} & 0 & 0 & 0 & 0 & \psi_{0t} \\ 0 & 0 & 0 & \rho_u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \\ u_{t-1} \\ w_{t-1} \\ E_{t-1}^{PA}(y_t) \\ E_{t-1}^{PA}(\pi_t) \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_t^u \\ \varepsilon_t^w \\ \omega_t^{PA} \end{pmatrix}$$

or

$$A^{PA} z_t^{PA} = B^{PA} E_t^{PA} \left(z_{t+1}^{PA} \right) + C_t^{PA} z_{t-1}^{PA} + D^{PA} \varepsilon_t^{PA}$$

The solution will take the form:

$$z_t^{PA} = \Gamma_{1,t} z_{t-1}^{PA} + \Gamma_{2,t} \varepsilon_t^{PA}$$

thus

$$E_t^{PA} \left(z_{t+1}^{PA} \right) = \Gamma_{1,t} z_t^{PA}$$

and we can rewrite

$$A^{PA} z_t^{PA} = B^{PA} \Gamma_{1,t} z_t^{PA} + C_t z_{t-1}^{PA} + D^{PA} \varepsilon_t^{PA}$$

This expression can be used to solve for $\Gamma_{1,t}$ and $\Gamma_{2,t}$; we use Sims'(2001) Gensys program for this purpose and we allow for the possibility of an indeterminate solution.

Matrices in the ALM

The ALM for the variables in the model can be obtained from the true equations (1) − (4) together with the true interest rate rule expressed by (13), and can be written in matrix form as:

$$\begin{pmatrix} 1 & 0 & \frac{1}{\sigma} & -\frac{1}{\sigma} & 0 & 0 & 0 & 0 & -\frac{1}{\sigma}\overline{r} \\ \frac{\kappa}{(1+\beta)} & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_t \\ \pi_t \\ i_t \\ u_t \\ w_t \\ v_t \\ E_t^{PA}(y_{t+1}) \\ E_t^{PA}(\pi_{t+1}) \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & \frac{1}{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta}{(1+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} E_t^{PA} \left[\begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \\ u_{t+1} \\ w_{t+1} \\ v_{t+1} \\ E_{t+1}^{PA}(y_{t+2}) \\ E_{t+1}^{PA}(\pi_{t+2}) \\ 1 \end{pmatrix} \right] +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_{yt} & \psi_{\pi t} & \psi_{it} & 0 & 0 & 0 & 0 & 0 & \psi_{0t} \\ 0 & 0 & 0 & \rho_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \\ u_{t-1} \\ w_{t-1} \\ v_{t-1} \\ E_{t-1}^{PA}(y_t) \\ E_{t-1}^{PA}(\pi_t) \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_t^u \\ \varepsilon_t^w \\ \varepsilon_t^v \end{pmatrix}$$

or:

$$Az_t = BE_t^{PA}\left(z_{t+1}^{PA}\right) + C_t z_{t-1} + D\varepsilon_t$$

Due to indeterminacy, the matrices of the ALM have an invertibility issue. To side-step this problem, we plug in the PLM of private agents twice to get:

$$
\begin{aligned}
E_t^{PA}\left(z_{t+1}^{PA}\right) &= \Gamma_{1,t} z_t^{PA} \\
&= \Gamma_{1,t}^2 z_{t-1}^{PA} + \Gamma_{1,t}\Gamma_{2,t}\varepsilon_t^{PA}
\end{aligned}
$$

and from this we can write:

$$
Az_t = B\left(\Gamma_{1,t}^2 M^{PA} z_{t-1} + \Gamma_{1,t}\Gamma_{2,t}\varepsilon_t^{PA}\right) + C_t z_{t-1} + D\varepsilon_t \tag{15}
$$

where M^{PA} is a matrix selecting the elements of z_t^{PA} from z_t. Notice that the elements of ε_t^{PA} can be written as a function of the elements of ε_t and z_{t-1}, since ε_t^u and ε_t^w are included in ε_t and

$$
\begin{aligned}
\omega_t^{PA} &= i_t - \psi_t z_{t-1}^R \\
&= -\left(F_t + \psi_t\right) z_{t-1}^R + v_t \\
&= -\left(F_t + \psi_t\right) z_{t-1}^R + \varepsilon_t^v
\end{aligned}
$$

where the last step follows from the simplifying assumption that we made in the implementation of our model that $\rho_v = 0$. Notice however that ω_t^{PA} enters in (15). For this reason, we adjust the model to account for private agents' perceived variance of ω_t^{PA} as follows:

$$
\begin{aligned}
\varepsilon_t^{PA} = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma_v/\widehat{\sigma}_{\omega t} \end{pmatrix} \varepsilon_t \\
&- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\widehat{\sigma}_{\omega t} & 0 \end{pmatrix} \left(F_t + \psi_t\right) z_{t-1}^R
\end{aligned}
$$

or:

$$
\begin{aligned}
\varepsilon_t^{PA} &= S_1\varepsilon_t + S_2 z_{t-1}^R \\
&\quad S_1\varepsilon_t + S_2 M^R z_{t-1}
\end{aligned}
$$

where $\widehat{\sigma}_{\omega t}$ is the estimated standard deviation of ω_t^{PA}, and M^R is a matrix selecting the elements of z_{t-1}^R from z_{t-1}. Finally, the ALM of the model can be written as:

$$
z_t = \Psi_{1,t} z_{t-1} + \Psi_{2,t}\varepsilon_t \tag{16}
$$

where

$$\Psi_{1,t} = A^{-1}\left(B\Gamma_{1,t}^2 M^{PA} + C + \Gamma_{1,t}\Gamma_{2,t}S_2 M^R\right)$$

$$\Psi_{2,t} = A^{-1}\left(D + \Gamma_{1,t}\Gamma_{2,t}S_1\right)$$

References

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